<u>Rmk</u>: Given $\gamma : [0,1] \longrightarrow S$ a regular curve with $\gamma(0) = p$ $\gamma(1) = q$, we get a bijective map

$(-\varepsilon, 1+\varepsilon)$

$$P_\gamma : T_p S \longrightarrow T_q S$$

Given by $P_\gamma(X_0) = X(1)$ where X is the unique parallel vector field along γ with $X(0) = X_0$. In fact, P_γ is a linear isometry: P_γ is a linear isomorphism with

$$\langle P_\gamma(X_0), P_\gamma(Y_0)\rangle_q = \langle X_0, Y_0\rangle_p$$

In this way, the covariant derivative ∇ gives us a way to relate "connect" the tangent spaces $(T_p S, \langle , \rangle_p)$ for different points $p \in S$. For this reason, it's also called a <u>connection</u>.

$$\left\{\begin{array}{c}\text{regular curves}\\ \gamma : [0,1] \to S\\ \gamma(0) = p,\ \gamma(1) = q\end{array}\right\} \xrightarrow{\nabla} \left\{\begin{array}{c}\text{linear isometries}\\ T_p S \xrightarrow{\sim} T_q S\end{array}\right\}$$

$$\gamma \longmapsto P_\gamma$$

in particular, if $p = q$

$$\left\{\begin{array}{c}\text{regular closed curves}\\ \gamma : [0,1] \to S\\ \gamma(0) = \gamma(1) = p\end{array}\right\} \xrightarrow{\text{Holonomy}} \begin{array}{c}\text{Orthogonal group of}\\ (T_p S, \langle , \rangle_p)\end{array}$$

$$\gamma \longmapsto P_\gamma : T_p S \longrightarrow T_p S$$

Def 1) A nonconstant parametrized curve $c : I \to S$ is a geodesic if $\nabla_{\dot c} \dot c = 0$, i.e. if the tangent vector field $\dot c$ is parallel along c.

2) a curve $C \subseteq S$ is a geodesic if locally around each point $p \in C$, $\exists$ a parametrizat $\gamma : I \to S$, $\gamma(t_0) = p$ s.t. γ is a geodesic.

Idea: In Euclidian space, $D_{\dot c} \dot c = \ddot c$ is the acceleration of c so $D_{\dot c} \dot c = 0 \iff$ c has zero acceleration $\iff$ c describes the motion of a particle with no force on it.

On a surface, we don't have D but we have ∇ so $\nabla_{\dot c} \dot c$ is simply the acceleration of c along the surface S (i.e. the tangent component of $\ddot c$)

Thus geodesics are the paths of a "force-free" particle (from the pov. of the surface S)

Fact The length $\|\dot c\|$ is constant for a geodesic curve c. In particular, a constant reparametrization is by arc length & $\|\dot c\| \neq 0$

Thm (Existence of geodesics)

let $p_0 \in S$, $Y_0 \in T_{p_0} S$ with $\|Y_0\| = 1$. Then there exists $\varepsilon > 0$ & a unique geodesic $c : (-\varepsilon, \varepsilon) \to S$ with $c(0) = p_0$, $\dot{c}(0) = Y_0$.

Pf pick a chart $f : U \to S$ with $f(u_0) = p_0$. We want $\gamma : (-\varepsilon, \varepsilon) \to U$ with $c = f \circ \gamma$ geodesic & ~~with~~ $\gamma(0) = u_0$

Write $\gamma = (u^1(t), \ldots, u^n(t))$

$$0 = \nabla_{\dot{c}} \dot{c} = \nabla_{\dot{u}^i \frac{\partial f}{\partial u^i}} \left(\dot{u}^j \frac{\partial f}{\partial u^j} \right) = \dot{u}^i \nabla_{\frac{\partial f}{\partial u^i}} \left(\dot{u}^j \frac{\partial f}{\partial u^j} \right)$$

$$= \dot{u}^i \left(\frac{\partial \dot{u}^k}{\partial u^i} \frac{\partial f}{\partial u^k} + \dot{u}^j \Gamma^k_{ij} \frac{\partial f}{\partial u^k} \right)$$

$$= \left(\ddot{u}^k + \dot{u}^i \dot{u}^j \Gamma^k_{ij} \right) \frac{\partial f}{\partial u^k}$$

So we need to solve the following

$\boxed{\text{geodesic equations}}$

$$\boxed{\begin{array}{c} \ddot{u}^k + \dot{u}^i \dot{u}^j \Gamma^k_{ij} = 0 \\ \text{for all } k = 1, \ldots, n \end{array}}$$

Recall c is arc-length

Rmk Note that by definition $\nabla_{\dot c}\dot c$ is the tangent component of $D_{\dot c}\dot c = \ddot c = \kappa n$
normal to c

i.e. $\|\nabla_{\dot c}\dot c\|$ is the geodesic curvature we defined before (up to sign):

Note everything is oriented: consider $\underbrace{N \times \dot c}$ unit tangent to S which is normal to $\dot c$

$$\nabla_{\dot c}\dot c = \kappa_g (N \times \dot c)$$

$$\kappa n = \kappa_n N + \kappa_g (N \times \dot c) \qquad \kappa^2 = \kappa_n^2 + \kappa_g^2$$

We see $\nabla_{\dot c}\dot c = 0 \iff \kappa_g = 0 \iff \pm\kappa = \kappa_n$

$\iff n$ is parallel to N, so this agrees with our previous notion.

Thm $p, q \in S$ fixed points & $c : I = [a,b] \to S$ C^∞ curve with $c(a) = p$, $c(b) = q$ minimizing length, then c is a geodesic, (up to reparametrization)

Pf Suppose c is an arc length curve with length L, $I = [0, L]$ and consider a family of curves $A(s,t)$ $s \in I$ $t \in [-\varepsilon, \varepsilon]$