

Rmk : Given  $\gamma : [0, 1] \rightarrow S$  a regular curve  
 with  $\gamma(0) = p$   $\gamma(1) = q$ , we get a bijective map

$$P_\gamma : T_p S \rightarrow T_q S$$

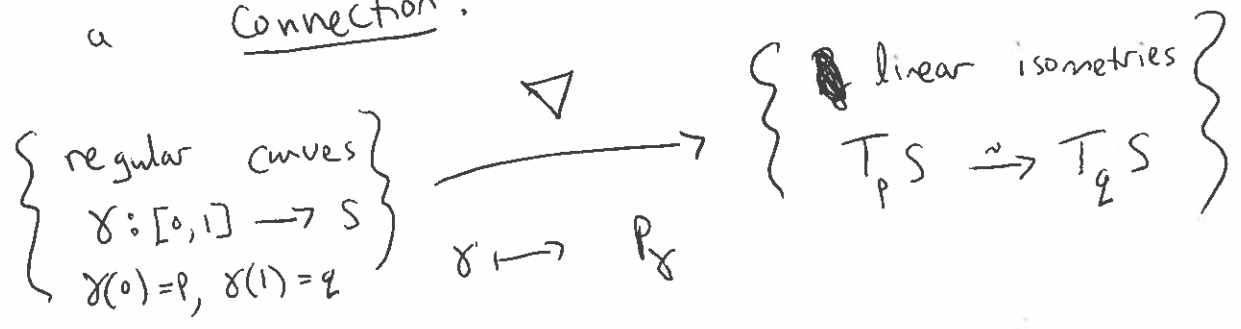
Given by  $P_\gamma(X_0) = X(1)$  where  $X$   
 is the unique parallel vector field along  $\gamma$  with

~~$X(0) = X_0$~~   $X(0) = X_0$ . In fact,  $P_\gamma$  is a linear

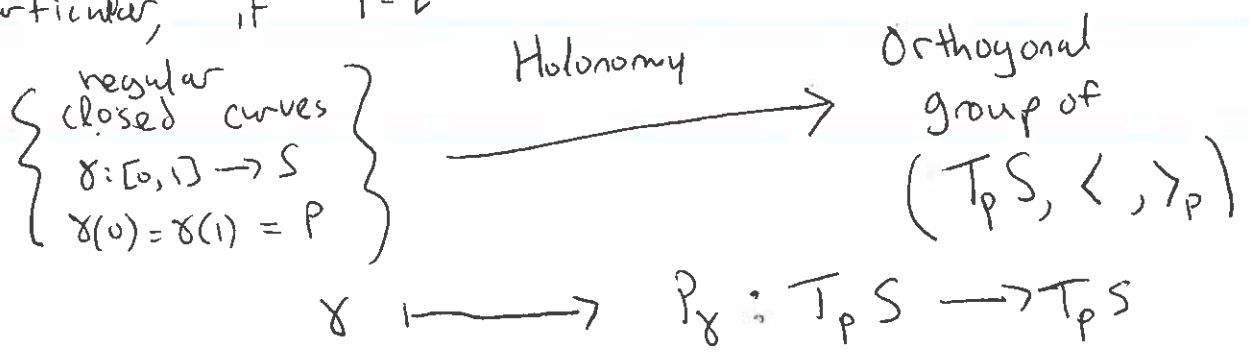
isometry:  $P_\gamma$  is a linear isomorphism with

$$\langle P_\gamma(X_0), P_\gamma(Y_0) \rangle_q = \langle X_0, Y_0 \rangle_p$$

In this way, the covariant derivative  $\nabla$  gives  
 us a way to relate the tangent spaces  $(T_p S, \langle \cdot, \cdot \rangle_p)$   
 for different points  $p \in S$ . For this reason, it is also  
 called a connection.



in particular, if  $p = q$



Def 1) A non constant <sup>parametrized</sup> curve  $c: I \rightarrow S$  is a geodesic if  $\nabla_{\dot{c}} \dot{c} = 0$ , i.e. if the tangent vector field  $\dot{c}$  is parallel along  $c$ .

2) a curve  $C \subseteq S$  is a geodesic if locally around each point  $p \in C$ ,  $\exists$  a parametrization  $\gamma: I \rightarrow S$ ,  $\gamma(t_0) = p$  s.t.  $\gamma$  is a geodesic.

Idea: In Euclidian space,  $D_{\dot{c}} \dot{c} = \ddot{c}$  is the acceleration of  $c$  so  $D_{\dot{c}} \dot{c} = 0 \iff c$  has zero acceleration  $\iff c$  describes the motion of a particle with no force on it.

On a surface, we don't have  $D$  but we have  $\nabla$  so  $\nabla_{\dot{c}} \dot{c}$  is simply the acceleration of  $c$  along the surface  $S$  (i.e. the tangent component of  $\ddot{c}$ )

Thus geodesics are the paths of a "force-free" particle (from the pov. of the surface  $S$ )

~~The~~ Fact The length  $\|\dot{c}\|$  is constant for a geodesic curve  $c$ . In particular, a constant reparametrization is by arc length &  $\|\dot{c}\| = 1$

# Thm (Existence of geodesics)

Let  $p_0 \in S$ ,  $Y_0 \in T_{p_0} S$  with  $\|Y_0\| = 1$ . Then there exists  $\epsilon > 0$  & a unique geodesic

$$c: (-\epsilon, \epsilon) \rightarrow S \text{ with } c(0) = p_0, \quad \dot{c}(0) = Y_0.$$

Pf pick a chart  $f: U \rightarrow S$  with

$$f(u_0) = p_0. \text{ We want } \gamma: (-\epsilon, \epsilon) \rightarrow U \text{ with}$$

$$\gamma(0) = u_0 \quad \& \quad c = f \circ \gamma \text{ geodesic}$$

Write  $\gamma = (u^1(t), \dots, u^n(t))$

$$0 = \nabla_{\dot{c}} \dot{c} = \nabla_{\dot{u}^i \frac{\partial f}{\partial u^i}} \left( \dot{u}^j \frac{\partial f}{\partial u^j} \right) = \dot{u}^i \nabla_{\frac{\partial f}{\partial u^i}} \left( \dot{u}^j \frac{\partial f}{\partial u^j} \right)$$

$$= \dot{u}^i \left( \frac{\partial \dot{u}^k}{\partial u^i} \frac{\partial f}{\partial u^k} + \dot{u}^j \Gamma_{ij}^k \frac{\partial f}{\partial u^k} \right)$$

$$= \left( \ddot{u}^k + \dot{u}^i \dot{u}^j \Gamma_{ij}^k \right) \frac{\partial f}{\partial u^k}$$

So we need to solve the following

geodesic equations

$$\ddot{u}^k + \dot{u}^i \dot{u}^j \Gamma_{ij}^k = 0$$

for all  $k=1, \dots, n$

Recall  $c$  is arc-length  
 Note that by definition  $\nabla_{\dot{c}} \dot{c}$  is  
 the tangent component of  $D_{\dot{c}} \dot{c} = \ddot{c} = \|K_n\|$   
 i.e.  $\|\nabla_{\dot{c}} \dot{c}\|$  is the geodesic curvature we defined before (up to sign):

Note everything is oriented: consider  $N \times \dot{c}$   
 unit tangent to  $S$  which is normal to  $\dot{c}$

$$\nabla_{\dot{c}} \dot{c} = K_g (N \times \dot{c})$$

$$K_n = K_n N + K_g (N \times \dot{c}) \quad K^2 = K_n^2 + K_g^2$$

We see  $\nabla_{\dot{c}} \dot{c} = 0 \iff K_g = 0 \iff \pm K = K_n$   
 $\iff n$  is parallel to  $N$ , so this agrees with our previous notion.

Thm  $p, q \in S$  fixed points &  $c: I=[a, b] \rightarrow S$   
 $C^\infty$  curve with  $c(a)=p, c(b)=q$  minimizing length,  
 then  $c$  is a geodesic, (up to reparametrization)

PF Suppose  $c$  is an arc length curve with length  $L, I=[0, L]$  and consider a family of curves  $A(s, t)$   $s \in I, t \in [-\epsilon, \epsilon]$