Recall c is arc-length

Rmk Note that by definition $\nabla_{\dot c}\dot c$ is the tangent component of $D_{\dot c}\dot c = \ddot c = \kappa n$

normal to c

i.e. $\|\nabla_{\dot c}\dot c\|$ is the geodesic curvature we defined before (up to sign):

Note everything is oriented: consider $\underbrace{N \times \dot c}$ unit tangent to S which is normal to $\dot c$

$$\nabla_{\dot c}\dot c = \kappa_g (N \times \dot c)$$

$$\kappa n = \kappa_n N + \kappa_g (N \times \dot c) \qquad \kappa^2 = \kappa_n^2 + \kappa_g^2$$

We see $\nabla_{\dot c}\dot c = 0 \iff \kappa_g = 0 \iff \pm\kappa = \kappa_n$

$\iff$ n is parallel to N, so this agrees with our previous notion.

Thm $p, q \in S$ fixed points & $c : I = [a,b] \to S$ C^∞ curve with $c(a) = p$, $c(b) = q$ minimizing length, then c is a geodesic, (up to reparametrization)

Pf Suppose c is an arc length curve with length L, $I = [0, L]$ and consider a family of curves $A(s,t)$ $\quad s \in I \quad t \in [-\varepsilon, \varepsilon]$

with $A(s,0) = c(s)$ $\quad$ $\left.\begin{aligned} A(0,t) &= c(0) \\ A(L,t) &= c(L) \end{aligned}\right\}$ fixed endpoint

$c_t(s) := A(s,t)$

$$L(t) = \mathrm{Length}(c_t) = \int_0^L \left\langle \frac{\partial c_t}{\partial s}, \frac{\partial c_t}{\partial s} \right\rangle^{1/2} ds$$

$$L'(t)\Big|_{t=0} = \frac{\partial}{\partial t}\Big|_{t=0} \int_0^L \left\langle \frac{\partial c_t}{\partial s}, \frac{\partial c_t}{\partial s} \right\rangle^{1/2} ds$$

$$= \int_0^L \frac{\partial}{\partial t}\Big|_{t=0} \left\langle \frac{\partial c_t}{\partial s}, \frac{\partial c_t}{\partial s} \right\rangle^{1/2} ds$$

$$= \int_0^L \frac{\left\langle \frac{\partial^2 A}{\partial t \partial s}, \frac{\partial A}{\partial s} \right\rangle\Big|_{t=0}}{\langle c', c' \rangle^{1/2}} ds$$

$$= \int_0^L \left\langle \frac{\partial^2 A}{\partial s \partial t}, \frac{\partial A}{\partial s} \right\rangle\Big|_{t=0} ds = \int_0^L \left\langle \nabla_{c'} \frac{\partial A}{\partial t}\Big|_{t=0}, c' \right\rangle ds$$

$$= \int_0^L \Big[\underbrace{\nabla_{c'}}_{\frac{\partial}{\partial s}} \left\langle \frac{\partial A}{\partial t}\Big|_{t=0}, c' \right\rangle - \left\langle \frac{\partial A}{\partial t}\Big|_{t=0}, \nabla_{c'} c' \right\rangle \Big] ds$$

$$\cancel{\left\langle \frac{\partial A}{\partial t}(0,L), c' \right\rangle}_{0} - \cancel{\left\langle \frac{\partial A}{\partial t}(0,0), c' \right\rangle}_{0} - \int_0^L \left\langle \frac{\partial A}{\partial t}\Big|_{t=0}, \nabla_{c'} c' \right\rangle ds$$

but if c has min length, $L'(0)=0$ for all A:

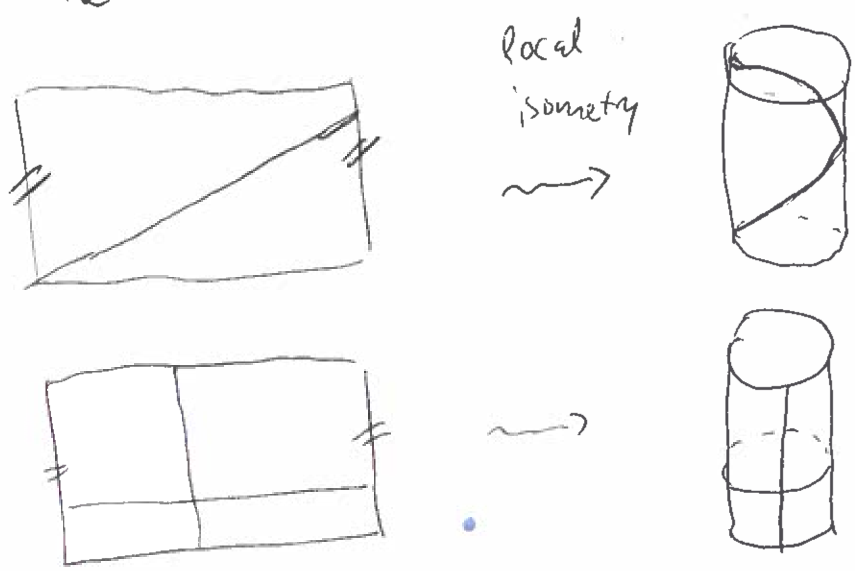
Thus, $\displaystyle\int_0^L \left\langle \frac{\partial A}{\partial t}\Big|_{t=0}, \nabla_{c'} c' \right\rangle ds = 0$ for all A, $\Rightarrow$ $\nabla_{c'} c' \equiv 0$ so c is geodesic.

$\boxtimes$

<u>Ex</u> 1) lines on $\mathbb{R}^n$ are geodesics. By uniqueness, there are the only ones

2) great circles on S^2 are geodesics since they're normal sections. By uniqueness, Only ones.

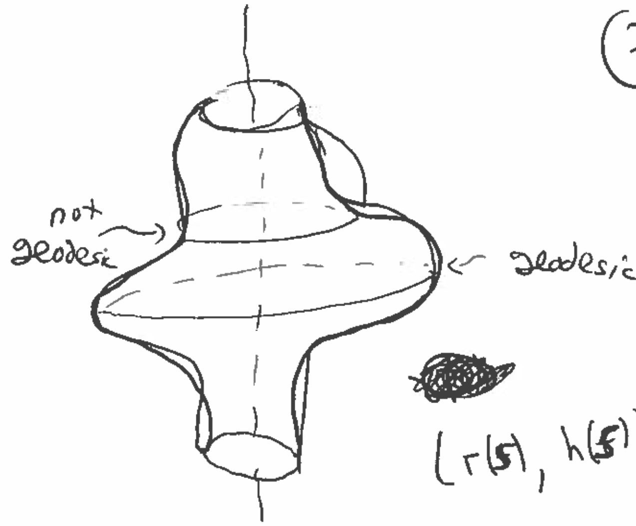
3)



<u>Fact</u> Being a geodesic is a local & intrinsic condition: geodesics are invariant under local isometries

4) Surfaces of revolution:

the constant $\theta = \theta_0$ curves are all geodesics



$$f(s,\theta) = (r\cos\theta, r\sin\theta, h)$$

Constant $s = s_0$ curves are geodesics $\iff$ the tangent plane is parallel to the z-axis $\iff$ $r'(s_0) = 0$

check this using the geodesic equations

$$\Gamma^1_{11} = \Gamma^2_{22} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = 0 \qquad \Gamma^1_{22} = -rr'$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{r'}{r}$$

$$\gamma(t) = (s(t), \theta(t))$$

$$\ddot{s}(t) + (\dot{\theta}(t))^2(-rr') = 0 \qquad \frac{d^2 s}{dt^2} = \left(\frac{d\theta}{dt}\right)^2 rr'$$

$$\ddot{\theta}(t) + 2\dot{\theta}\dot{s}\frac{r'}{r} = 0 \qquad \frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{ds}{dt}\frac{r'}{r} = 0$$

We see if $\theta = \theta_0$ constant can solve the equs,

& if $s = s_0$ is constant, we can solve $\iff r' = 0$