Taylor expansion

$$c(s) = c(0) + c'(0)s + c''(0)\frac{s^2}{2} + \dots + c^{(n)}(0)\frac{s^n}{n!} + \dots$$

**Def** $c_1$ & $c_2$ have contact order $k$

if $c_1^{(n)}(0) = c_2^{(n)}(0)$ for $n = 0, \dots, k$

e.g. if $k = 1$ then $c_1$ & $c_2$ are tangent

**Def** (Frenet frames & curves)

let $c: I \to \mathbb{R}^n$ be a regular $C^n$-curve par by arc length

$c$ is <u>Frenet</u> if $c', \dots, c^{(n-1)}$ are linearly independent. In this case, a <u>Frenet frame</u> $e_1, \dots, e_n$ is an orthonormal positively oriented basis of $T_{c(t)}\mathbb{R}^n$ (depending on $t$!) s.t.

1) $\mathrm{Span}(e_1, \dots, e_k) = \mathrm{Span}(c', c'', \dots, c^{(k)})$

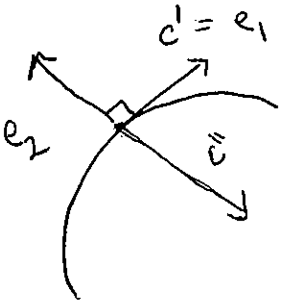
for all $k = 1, \dots, n-1$

2) $\langle c^{(k)}, e_k \rangle > 0$ for $k = 1, \dots, n-1$

given $c', \dots, c^{(n-1)}$ $\xrightarrow{\text{gram schmidt}}$ $e_1, \dots, e_{n-1}$

Then $e_n$ is uniquely determined by the positive orientation.

<u>Ex</u> 1) for $n=2$, every regular $C^2$ curve is Frenet




$e_1$ = unit tangent

$e_2$ = unit normal

$$0 = \frac{d}{ds}\langle c', c'\rangle = 2\langle c', c''\rangle = 2\langle e_1, c''\rangle$$

So $c'' = \kappa e_2$ for some function $\kappa$ called the <u>curvature</u>

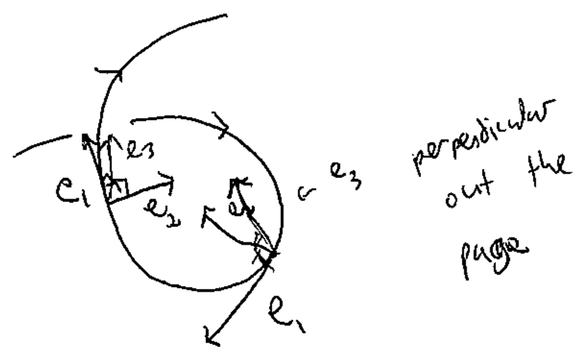
2) for $n=3$ a regular $C^3$ curve is Frenet

if $c'' \neq 0$

$e_1 = c'$

$e_2 = \dfrac{c''}{\|c''\|}$

$e_3 = e_1 \times e_2$

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<u>Plane curves</u>

$$e_1' = c'' = \kappa e_2$$

$$e_2' = -\kappa e_1$$

$$c'' = \kappa e_2$$

$$0 = \frac{d}{ds}\langle e_1, e_2\rangle$$

use this to compute the $2^{nd}$ eq$^n$

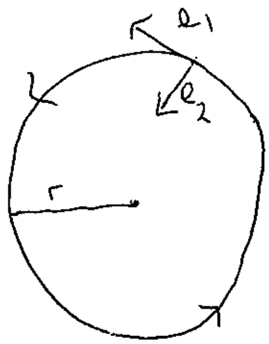
$$\frac{d}{ds}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \text{Frenet eqns}$$

Ex 1)

$$c(s) = \left( r\cos\left(\frac{s}{r}\right),\ r\sin\left(\frac{s}{r}\right) \right)$$

$$c'(s) = \left( -\sin\left(\frac{s}{r}\right),\ \cos\left(\frac{s}{r}\right) \right)$$

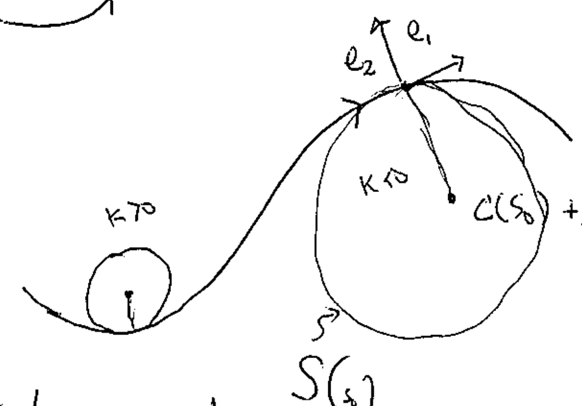
$$c''(s) = \frac{-1}{r}\left( \cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right) \right)$$




$$= \frac{1}{r} e_2 \qquad \kappa = 1/r \quad \text{is constant}$$

if $\kappa(s_0) \neq 0$

2)




$$c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) =: z(s_0)$$

Osculating circle

Centered at $z(s_0)$

radius $\frac{1}{|\kappa(s_0)|}$

$S(s_0)$ meets $c(s_0)$ to order 2

3) Consider a graph $y = f(x)$ & suppose $f(0) = f'(0) = 0$, $f''(0) \neq 0$

$$c(t) = (t, f(t)) \qquad \dot{c}(t) = (1, f'(t)) \qquad e_1 = \dot{c}(0) = (1, 0) \text{ unit vector}$$

$$\kappa(0) = f''(0) \Longleftarrow \ddot{c}(0) = (0, f''(0)) = f''(0)(0,1) = f''(0)\, e_2$$

**Thm** A regular plane curve has constant curvature $\kappa$ if & only if either

1) it is an arc of a circle of radius $\frac{1}{|\kappa|}$ (if $\kappa \neq 0$)

2) it is a line segment if $\kappa = 0$

**Proof** Suppose $\kappa \neq 0$, ~~then the system~~

~~has a unique solution~~

Consider the osculating circle with center

$$z(s) = c(s) + \frac{1}{\kappa} e_2(s)$$

& constant radius $\frac{1}{\kappa}$

$$z'(s) = c'(s) + \frac{1}{\kappa} e_2'(s) = e_1(s) + \frac{1}{\kappa}\left(-\kappa e_1(s)\right) = 0$$

Frenet eqns

thus $z(s) = z_0$ is constant. Moreover,

$$\| z(s) - c(s) \| = \frac{1}{\kappa}$$

so $c(s)$ lies on the circle $S\left(z_0, \frac{1}{\kappa}\right)$ for all $s$.

If $K=0$, then by Frenet eqns

$c^{(n)}=0$ for all $n \geq 2$ &

$c'(s) = (a_1, a_2) = \vec{a}$ so $c(s) = \vec{a}\, s + \vec{b}$
is a line, $\square$

Thm For each continuous function $K(s)$, there exists a regular plane curve unique up to translation & rotation, ~~that~~ with curvature $K$.

Pf idea

Write $e_1 = (\cos(\alpha(s)), \sin(\alpha(s)))$, by orthonormality,
$e_2 = (-\sin(\alpha(s)), \cos(\alpha(s)))$. $Ke_2 = e_1' = \alpha' e_2$

So $K = \alpha'(s)$ Suppose wlog $c(0)=(0,0)$
$e_1(0) = (1,0)$
then $\alpha(0)=0$, $\alpha(s) = \int_0^s K(t)\,dt$ by FTC

Now $x(s) = \int_0^s \cos\left(\int_0^\sigma K(t)\,dt\right) d\sigma$ $y(s) = \int_0^s \sin\left(\int_0^\sigma K(t)\,dt\right) d\sigma$

$c(s) = (x(s), y(s))$ & $c(s)$ is unique by the theory of DE $\square$

# Space Curves $(n=3)$

a regular $C^3$ - curve is Frenet iff $c'' \neq 0$

$e_1 = c'$ unit tangent

$e_2 = \dfrac{c''}{\|c''\|}$ principal normal

$e_3 = e_1 \times e_2$ binormal

$K := \|c''\|$ is the curvature

$e_1' = K e_2,$

$$e_2' = \underbrace{\langle e_2', e_1\rangle}_{=\langle -e_2, e_1'\rangle = -K} e_1 + \underbrace{\langle e_2', e_2\rangle}_{=0} e_2 + \underbrace{\langle e_2', e_3\rangle}_{=:\tau} e_3$$

$$= -K e_1 + \tau e_3$$

$$e_3' = \underbrace{\langle e_3', e_1\rangle}_{=\langle e_3, -e_1'\rangle = 0} e_1 + \underbrace{\langle e_3', e_2\rangle}_{=\langle e_3, -e_2'\rangle = -\tau} e_2 + \underbrace{\langle e_3', e_3\rangle}_{=0} e_3$$

$$= -\tau e_2$$

$\tau$ is the torsion

$$\frac{d}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$\boxed{\text{note planar} \iff \tau = 0!}$