Lemma let X, Y, Z vector fields on an open subset of $\mathbb{R}^n$, then

$$D_X D_Y Z - D_Y D_X Z = D_{[X,Y]} Z$$

$$\text{"} [D_X, D_Y] = D_{[X,Y]} \text{"}$$

Pf

if $X = e_i$ $Y = e_j$, $[e_i, e_j] = 0$

$$D_{e_i} D_{e_j} = \frac{\partial^2}{\partial u^i \partial u^j} = \frac{\partial^2}{\partial u^j \partial u^i} = D_{e_j} D_{e_i} \quad \checkmark$$

$$X = \xi^i e_i \qquad Y = \eta^j e_j$$

$$[D_X, D_Y] Z = \xi^i D_{e_i}(\eta^j D_{e_j} Z) - \eta^i D_{e_i}(\xi^j D_{e_j} Z)$$

$$= \underbrace{\cancel{\xi^i \eta^j D_{e_i} D_{e_j} Z - \eta^i \xi^j D_{e_i} D_{e_j} Z}}_{0} + \xi^i \frac{\partial \eta^j}{\partial u^i} D_{e_j} Z - \eta^i \frac{\partial \eta^j}{\partial u^i} D_{e_j} Z$$

$$= \left(\xi^i \frac{\partial \eta^j}{\partial u^i} - \eta^i \frac{\partial \xi^j}{\partial u^i} \right) D_{e_i} Z = D_{[X,Y]} Z$$

Thm (Coordinate free Gauss & Codazzi-Mainardi:)

X, Y, Z tangent vector fields to S along a chart $f : U \to S \subseteq \mathbb{R}^{n+1}$, then

i) $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \mathrm{II}(Y,Z) LX - \mathrm{II}(X,Z) LY$

iii) $\nabla_X(LY) - \nabla_Y(LX) - L([X,Y]) = 0$

Pf $D_X Y = \nabla_X Y + \mathrm{II}(X,Y)\nu$ $\qquad \nu = \text{normal}$

$$0 = [D_X, D_Y]Z - D_{[X,Y]}Z =$$

$$D_X(\nabla_Y Z + \mathrm{II}(Y,Z)\nu) - D_Y(\nabla_X Z + \mathrm{II}(X,Z)\nu)$$

$$- \nabla_{[X,Y]}Z - \langle L([X,Y]), Z\rangle \nu$$

$$= \nabla_X\nabla_Y Z + \mathrm{II}(X, \nabla_Y Z)\nu + D_X \mathrm{II}(Y,Z)\nu + \mathrm{II}(Y,Z) D_X \nu$$

$$- \nabla_Y \nabla_X Z - \mathrm{II}(Y, \nabla_X Z)\nu - D_Y \mathrm{II}(X,Z)\nu - \mathrm{II}(X,Z) D_Y \nu$$

$$- \nabla_{[X,Y]}Z - \langle L([X,Y]), Z\rangle \nu$$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z + \mathrm{II}(Y,Z)LX$$
$$- \mathrm{II}(X,Z)LY$$
$$+ (\qquad\qquad\qquad)\nu$$

Def (Riemannian curvature ~~tensor~~) for tangent vector fields X, Y, Z

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

By the theorem, $R(X,Y)Z$ is a vector field which depends only on the values of X, Y, Z at p b/c the RHS of Gauss only depends on the values:

$$R(X,Y)Z = \langle LY, Z\rangle LX - \langle LX, Z\rangle LY \qquad (*)$$
$$= \mathrm{II}(Y,Z)\,LX - \mathrm{II}(X,Z)\,LY$$

$R(X,Y)Z = 0$ if S is an open subset of $\mathbb{R}^n$ so $R(X,Y)Z$ measures the failure of S to be locally isometric to $\mathbb{R}^n$

Properties of R:

1) $R(\alpha^1 X_1 + \alpha^2 X_2, Y) = \alpha^1 R(X_1, Y) + \alpha^2 R(X_2, Y)$
$R(X, \beta^1 Y_1 + \beta^2 Y_2) = \beta^1 R(X, Y_1) + \beta^2 R(X, Y_2)$

2) $R(X,Y)(\gamma^1 Z_1 + \gamma^2 Z_2) = \gamma^1 R(X,Y)Z_1 + \gamma^2 R(X,Y)Z_2$

Both follow by $(*)$

In particular, to compute R, we can write vectors in terms of an orthonormal basis

Thm (Coordinate free Theorem Egrigium)

a) let X, Y orthornormal tangent vector fields defined on $U \subseteq S$ open, then
a surface

$$\langle R(X,Y)Y, X\rangle = K$$

b) more generally if $S \subseteq \mathbb{R}^{n+1}$ is a hypersurface & $X_1, \ldots, X_n$ are orthonormal vector fields of principal directions with curvature K_i (recall this means $LX_i = K_i X_i$)

then $\langle R(X_i, X_j) X_j, X_i \rangle = K_i K_j$

for $i \neq j$

Pf a) X, Y are an orthonormal basis for TS, so we can write L as $\begin{bmatrix} \langle LX, X\rangle & \langle LX, Y\rangle \\ \langle LY, X\rangle & \langle LY, Y\rangle \end{bmatrix}$ in this basis

Now, Gauss eqn becomes

$$\langle R(X,Y)Y, X\rangle = \langle LY, Y\rangle\langle LX, X\rangle - \langle LX, Y\rangle\langle LY, X\rangle = \det(L) = K$$