Thm (Coordinate free Theorem Egrigium)

a) let X, Y orthonormal tangent vector fields defined on $U \subseteq S$ open, then
a surface

$$\langle R(X,Y)Y, X\rangle = K$$

b) more generally if $S \subseteq \mathbb{R}^{n+1}$ is a hypersurface & $X_1, \ldots, X_n$ are orthonormal vector fields of principal directions with curvature K_i (recall this means $LX_i = K_i X_i$)

then $\langle R(X_i, X_j) X_j, X_i \rangle = K_i K_j$

for $i \neq j$

Pf a) X, Y are an orthonormal basis for TS, so we can write L as

$$\begin{bmatrix} \langle LX, X\rangle & \langle LX, Y\rangle \\ \langle LY, X\rangle & \langle LY, Y\rangle \end{bmatrix}$$

in this basis

Now, Gauss eqn becomes

$$\langle R(X,Y)Y, X\rangle = \langle LY, Y\rangle\langle LX, X\rangle - \langle LX, Y\rangle\langle LY, X\rangle = \det(L) = K$$

b) by assumption, $\langle X_i, X_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

$LX_i = K_i X_i$, then by Gauss

$$\langle R(X_i, X_j) X_j, X_i \rangle =$$

$$\langle LX_j, X_j \rangle \langle LX_i, X_i \rangle - \langle LX_i, X_j \rangle \langle LX_j, X_i \rangle$$

$$= K_j \underbrace{\langle X_j, X_j \rangle}_{1} K_i \underbrace{\langle X_i, X_i \rangle}_{1} - K_i \underbrace{\langle X_i, X_j \rangle}_{0} K_j \underbrace{\langle X_j, X_i \rangle}_{0}$$

$$= K_i K_j$$

<u>Cor</u> the quantities $K_i K_j$ are intrinsic

<u>Rmk</u> $K_i K_j$ is called the <u>sectional curvature</u> of the plane spanned by X_i & X_j : $\sigma = \mathrm{Span}(X_i, X_j) \subseteq T_p S$

For each plane $\sigma \subseteq T_p S$ we can define sectional curvature

$$K(\sigma, p) := \langle R(X, Y) Y, X \rangle \text{ where}$$

X, Y is an <u>orthonormal basis</u> for σ.

Rmk (geometric interpretation of R)

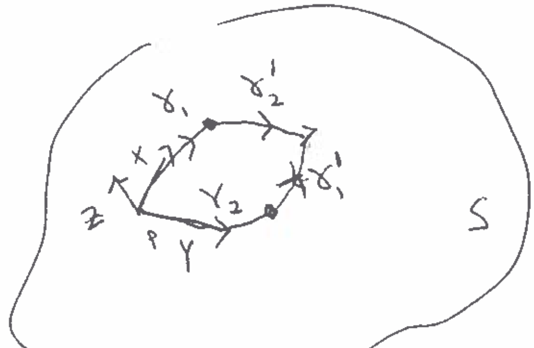
1) Via sectional curvature: given a plane $\sigma \subseteq T_p S$ we can integrate the vectors in σ into geodesics to form a surface $p \in H \subseteq S$ with $T_p H = \sigma \subseteq T_p S$ & $T_q H$ = parallel transport of σ along geodesic for $q \neq p$, then H has a metric by restriction from S & $K(\sigma, p) = K(H)_p$ the Gauss curvature of H at p. (This is Riemann's original definition)

2) Via parallel transport:

$\boxed{\text{Suppose } [X,Y]=0}$

parallel transport depends on path; this dependance $R(X,Y)Z$ measures

γ_1 & γ_2 are s-time geodesics with tangent ~~γ_1~~ X, Y respectively, γ_1' & γ_2' are t-time geodesics with tangent $P_{\gamma_2} X$ & $P_{\gamma_1} Y$ resp. Then



$$R(X,Y)Z = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} P_{\gamma_1}^{-1} P_{\gamma_2'}^{-1} P_{\gamma_1'} P_{\gamma_2}$$