<u>Thm</u> (Fundamental Theorem of the local theory of surfaces)

Suppose $U \subseteq \mathbb{R}^n$ open, g_{ij}, h_{ij} given smooth functions on U for $1 \le i,j \le n$ satisfying the Gauss eqns & Codazzi-Mainardi eqns. let $u_0 \in U$, $P_0 \in \mathbb{R}^{n+1}$, $X_i^{(0)} \in T_{P_0}\mathbb{R}^{n+1}$ $i=1,\ldots,n$ given initial conditions with $g_{ij}(u_0) = \langle X_i^{(0)}, X_j^{(0)} \rangle$ & $\nu^{(0)}$ a unit normal to $X_i^{(0)}$. Then there exists an open connected $P_0 \in V \subseteq U$ and a unique regular surface element

$$f : V \longrightarrow \mathbb{R}^{n+1}$$

with gauss map ν s.t.

1) $f(u_0) = P_0$

2) $\frac{\partial f}{\partial u^i}(u_0) = X_i^{(0)}$

3) $\nu(u_0) = \nu^{(0)}$

4) I & II of f are (g_{ij}) & (h_{ij}) respectively

<u>PF sketch</u> we have the Gauss & Weingarten formulas

$$\frac{\partial X_j}{\partial u^i} = \Gamma_{ij}^{k} X_k + h_{ij}\nu$$

$$\frac{\partial \nu}{\partial u^i} = -h_{ij}\, g^{jk} X_k$$

Want f s.t. $\frac{\partial f}{\partial u^i} = X_i$ so we better have

$$\frac{\partial^2 X_i}{\partial u^j \partial u^k} = \frac{\partial^2 X_i}{\partial u^k \partial u^j} \qquad \frac{\partial^2 \nu}{\partial u^i \partial u^j} = \frac{\partial^2 \nu}{\partial u^j \partial u^i}$$

but this is exactly Gauss + Codazzi - Mainardi eqns

Frobenius integrability theorem $\Rightarrow$ we can integrate these with initial condition $X_i^{(0)}, \nu^{(0)}$ to get

$$X_i, \nu : V \longrightarrow \mathbb{R}^n$$

$$\cap$$

$$U$$

Now we check that $\langle \nu, \nu \rangle = 1 \qquad \langle \nu, X_i \rangle = 0$

$$\langle X_i, X_j \rangle = g_{ij}$$

$$\frac{\partial \langle \nu, \nu \rangle}{\partial u^i} = 2 \left\langle \frac{\partial \nu}{\partial u^i}, \nu \right\rangle = -2 h_{ik} g^{kl} \langle X_l, \nu \rangle$$

$$\frac{\partial}{\partial u^i} \langle \nu, X_j \rangle = -h_{ik} g^{kl} \langle X_l, X_j \rangle + \Gamma_{ij}^k \langle \nu, X_k \rangle + h_{ij} \langle \nu, \nu \rangle$$

$$\frac{\partial}{\partial u^k} \langle X_i, X_j \rangle = \Gamma_{ik}^r \langle X_r, X_j \rangle + \Gamma_{jk}^s \langle X_i, X_s \rangle + h_{ik} \langle \nu, X_j \rangle + h_{jk} \langle X_i, \nu \rangle$$

but $\langle \nu, \nu \rangle = 1 \qquad \langle \nu, X_j \rangle = 0 \qquad \langle X_i, X_j \rangle = g_{ij}$

are solutions to these eqns, so by uniqueness, they're the only ones

Now we have x_i & ν, we want

f s.t. $\dfrac{\partial f}{\partial u_i} = x_i \qquad \nu = \dfrac{\partial f}{\partial u_i} \times \dfrac{\partial f}{\partial u_j} \qquad (**)$

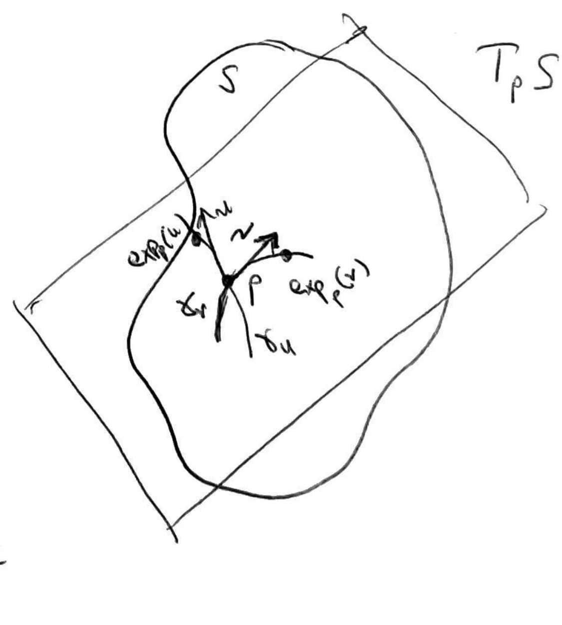
For this, we need integrability:

$$\frac{\partial x_i}{\partial u^j} = \Gamma^k_{ij} x_k + h_{ij}\nu = \Gamma^k_{ji} x_k + h_{ji}\nu = \frac{\partial x_j}{\partial u^i} \quad \checkmark$$

So we again apply Frobenius integrability to get f satisfying $(**)$ $\square$

The exponential map

$\gamma(t, v)$ = geodesic with
$\gamma(0) = p \quad \gamma'(0) = v$
$t \in (-\varepsilon, \varepsilon)$



Note $\gamma(t, \lambda v) = \underset{\overline{\gamma}}{\overset{\shortparallel}{\gamma(\lambda t, v)}}$

Indeed, $\overline{\gamma}' = \lambda \gamma'$ by chain rule

So 1) $\overline{\gamma}'(0) = \lambda v$ 2) $\overline{\gamma}(0) = p$

3) $\nabla_{\overline{\gamma}'} \overline{\gamma}' = \nabla_{\lambda\gamma'} \lambda\gamma' = \lambda^2 \nabla_{\gamma'} \gamma' = 0$

So $\overline{\gamma} = \gamma(t, \lambda v)$ by uniqueness of geodesics

We want to define a map

$$T_pS \longrightarrow S \quad \text{by taking}$$

$$v \longmapsto \gamma(1, v)$$

Call this the exponential map $\exp_p(v) := \gamma(1, v)$

$$\exp_p : \underset{T_pS}{\overset{U}{\cap\!\!\!\mid}} \longrightarrow S$$

<u>Prop</u> Given $p \in S$, there exists $\varepsilon > 0$ s.t. $\exp_p$ is well defined, differentiable, and a diffeomorphism on the ball

$$B_\varepsilon := \{ v \in T_pS \mid \|v\| < \varepsilon \} \subseteq T_pS.$$

<u>Pf Sketch</u> We know for each v, $\exists$ $\varepsilon(v)$ s.t. $\gamma(t, v)$ is well defined. Take $\|v\| = 1$
is $t \in (-\varepsilon(v), \varepsilon(v))$

Then $\gamma(t, v) = \gamma(1, tv) = \exp_p(tv)$ for all $t \in (-\varepsilon(v), \varepsilon(v))$ is well defined, & differentiable, we just need to know that $\varepsilon(v) > 0$ can be chosen small enough to be independent of v

This shows $\exp_p$ is well defined & diff on some ball B_ε for $\varepsilon > 0$ small enough

$$\exp_p : \underset{\substack{\cup \\ 0}}{B_\varepsilon^{\subseteq T_pS}} \longrightarrow S$$

What is $d(\exp_p)_0 : T_{\mathbf{0}} B_\varepsilon \longrightarrow T_p S$?

$$T_{\mathbf{0}} B_\varepsilon = T_p S$$

Let $\alpha_v(t) = tv$ be a curve in B_ε

$$\alpha_v'(0) = v \in T_p S$$

$$d(\exp_p)_0(v) = \frac{d}{dt} \exp_p(\alpha_v(t))\Big|_{t=0} = \frac{d}{dt} \exp_p(tv)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma(1, tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma(t, v) = v$$

Thus, $d(\exp_p)_0$ is the identity, which in particular is invertible so by Inverse function theorem, $\exp_p$ is invertible after possibly shrinking B_ε. ∎

Thus, $\exp_p$ gives us a coordinate chart at $p \in S$, in fact two distinguished coordinate charts

$\dim S = 2$ | e_1, e_2 orthonormal basis for T_pS | (3a)

Geodesic normal coordinates:

take u, v to be standard orthonormal coordinates on T_pS, then $f(u,v) := \exp_p(u e_1 + v e_2)$ is a chart with $(g_{ij})_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Geodesic polar coordinates

let r, θ be polar coordinates on T_pS

then $f(r,\theta) = \exp_p(r\cos\theta\, e_1 + r\sin\theta\, e_2)$

are coordinates with

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & G(r,\theta) \end{bmatrix}$$

Thm (Minding) Two surfaces with the same constant Gaussian curvature are locally isometric.

let $S, \tilde{S}$ two such surfaces with $p \in S$, $\tilde{p} \in \tilde{S}$

Pick orthonormal bases $\{e_1, e_2\}$ for T_pS $\{\tilde{e}_1, \tilde{e}_2\}$ for $T_{\tilde{p}}\tilde{S}$

In geodesic polar coordinates, $K = -\dfrac{\frac{\partial^2 \sqrt{G}}{\partial r^2}}{\sqrt{G}} = \text{Constant}$

So $\dfrac{\partial^2}{\partial r^2}\sqrt{G} + K\sqrt{G} = 0$ $\qquad$ $\dfrac{\partial^2}{\partial r^2}\sqrt{\tilde{G}} + K\sqrt{\tilde{G}} = 0$

By uniqueness of solutions to diff eq, $G = \tilde{G}$

thus $g_{ij} = \tilde{g}_{ij}$ so S & $\tilde{S}$ are isometric