Gauss-Bonnet Theorem

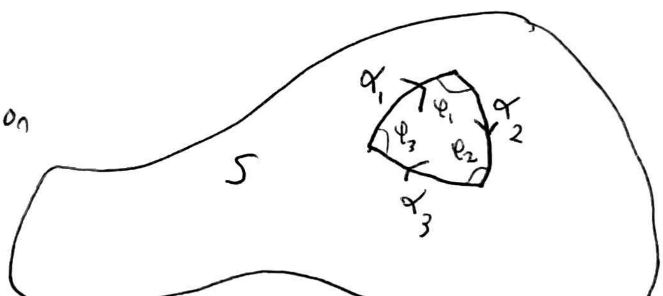
$\boxed{\text{Parametrized by arc length}}$

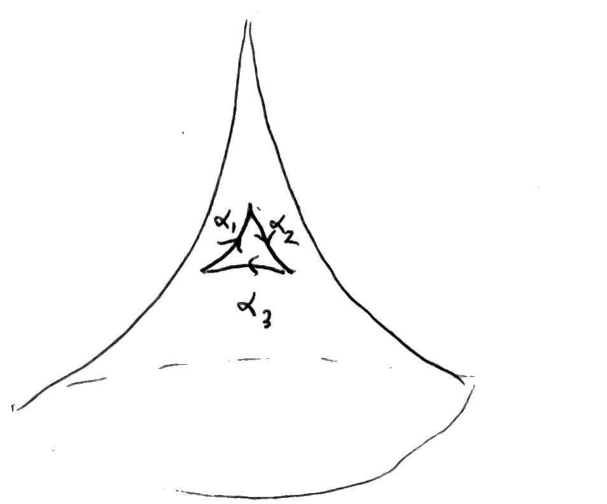
Def: A <u>geodesic triangle</u> is a simple, closed, continuous piecewise regular curve $\alpha : [a,b] \to S$ ~~such that~~ such that there exist exactly 2 times $t_1, t_2 \in [a,b]$ where α is not regular & $\alpha_1 = \alpha|_{[a,t_1]}$, $\alpha|_{[t_1,t_2]} = \alpha_2$ & $\alpha|_{[t_2,b]} = \alpha_3$ are geodesics. More generally, a <u>geodesic polygon</u> or <u>n-gon</u> is the same definition with $t_1, \dots, t_{n-1} \in (a,b)$ times where α is not regular.

Equivalently, it's a collection of n geodesics $\alpha_1, \dots, \alpha_n$ s.t.

$\alpha_i : [t_{i-1}, t_i]$, $\alpha_i(t_i) = \alpha_{i+1}(t_i)$

$$\alpha_1(t_0) = \alpha_n(t_n)$$





Def The <u>interior angles</u> φ_i & <u>exterior angle</u> θ_i are as in the picture:

$\theta_i \in (-\pi, \pi)$ $\quad \varphi_i \in (0, 2\pi)$

$\varphi_i = \pi - \theta_i$

Thm (Gauss-Bonnet I)

let $T \subseteq S$ be the interior of a geodesic triangle with interior angles $\varphi_1, \varphi_2, \varphi_3$.

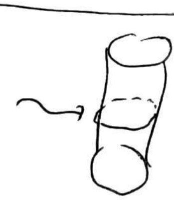
Then $$\varphi_1 + \varphi_2 + \varphi_3 - \pi = \iint_T K \, dA$$

K = Gauss Curvature

Rmk to make this precise, we need to assume T is contained in the image of a chart $f : U \to S$. So from now on, whenever we consider ~~the~~ the interior of a geodesic polygon or curve, we assume its contained in a chart.

e.g. don't consider curves like this

or



Cor 1) if $K > 0, \Rightarrow \sum \varphi_i > \pi$

2) if $K = 0, \Rightarrow \sum \varphi_i = 0$

3) if $K < 0, \Rightarrow \sum \varphi_i < 0$

Thm (Gauss-Bonnet II)

let $P \subseteq S$ be the interior of a geodesic n-gon with exterior angles $\theta_1, \dots, \theta_n$. Then

$$\iint_P K \, dA = 2\pi - \sum_{i=1}^{n} \theta_i$$

Example no bigons if $K < 0$

2-gons

More generally, we can consider piecewise regular simple closed curves $\alpha : [a,b] \to S$ which are not regular at <u>corners</u> $t_i \in [a,b]$, so same picture as before except the arcs α_i don't have to be geodesics,

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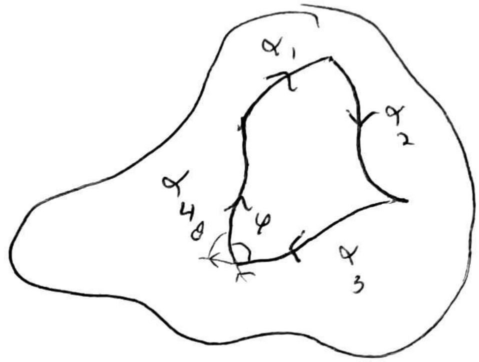
We have interior & exterior angles φ_i, θ_i as before



(Gauss-Bonnet III, local GB)

<u>Thm</u> let α be a piecewise regular closed simple curve with exterior angles θ_i & interior P contained in a chart of S. Then

$$\iint_P K\,dA + \int_\alpha K_g\,ds + \sum \theta_i = 2\pi$$

K_g = geodesic curvature

Note GB III $\Rightarrow$ GB II $\Rightarrow$ GB I

Indeed, α is a geodesic n-gon $\iff$ $K_g = 0$ for all $t \neq t_0, \dots, t_n$.

<u>Cor 1</u> for a plane simple closed curve $\alpha : I \to \mathbb{R}^2$, we have

$$\int_\alpha K\,ds + \sum \theta_i = 2\pi$$

<u>Cor 2</u> if α is a regular simple closed curve with no corners, then $\iint_P K\,dA + \int_\alpha K_g\,ds = 2\pi$

<u>Cor 3</u> For a smooth regular simple closed planar curve, then $\int_\alpha K_g\,ds = 2\pi$

Global version: What if we integrate K on all of S?

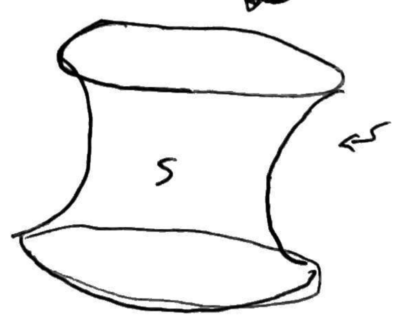
To make sense of this, we need S to be <u>Compact</u> so the integral converges.
" closed & bounded

For simplicity, we also want to avoid S having a boundary:

no <u>boundary</u>
good.

So we assume

S does <u>not</u> have a boundary.



has a <u>boundary</u>
bad

"S closes in on itself"

or "S is <u>closed</u>"

<u>Warning</u> this closed is a different meaning than above.

Thm (Gauss - Bonnet IV, global version)

let $S \subseteq \mathbb{R}^3$ be a compact oriented regular surface without boundary. Then

$$\iint_S K \, dA = 2\pi \, \chi(M)$$

where $\chi(M) \in \mathbb{Z}$ is the topological Euler characteristic.