Thm (Gauss - Bonnet IV, global version)

Let $S \subseteq \mathbb{R}^3$ be a compact oriented regular surface without boundary. Then

$$\iint_S K \, dA = 2\pi \, \chi(S)$$

Where $\chi(S) \in \mathbb{Z}$ is the topological Euler characteristic.

Lightning Review of $\chi(M)$

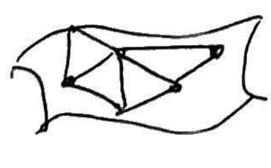
$f: S \to \bar{S}$ is a homeomorphism if its continuous with continuous inverse

a topological invariant is something which is the same for homeomorphic hypersurfaces

Claim $\chi(S) \in \mathbb{Z}$ is a topological invariant

What is $\chi(S)$? Compute via triangulation:

1) Cut S into triangles



2) $\chi(M) = V - E + F$

where $V = \#$ vertices, $E = \#$ of edges

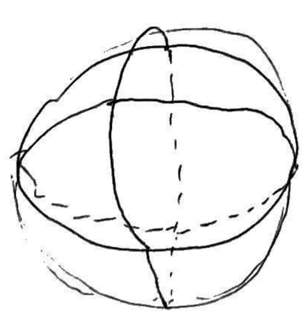
$F = \#$ faces/triangles in the triangulation.

<u>Claim</u> $V - E + F$ is independent of choice of triangulations

<u>Proof sketch</u> any two triangulations are related by refining, subdividing & edge flips. Check $V - E + F$ doesn't change after all these moves.

<u>Upshot</u> $\chi(S) \in \mathbb{Z}$ is a topological invariant. Thus GB IV says that $\int_S K \, dA$ is a topological invariant of S even though K very much is not!

<u>Ex</u> <u>Sphere</u>:



$\chi(S^2) = 2 - 4 + 4 = 2$

$= 6 - 12 + 8 = 2$

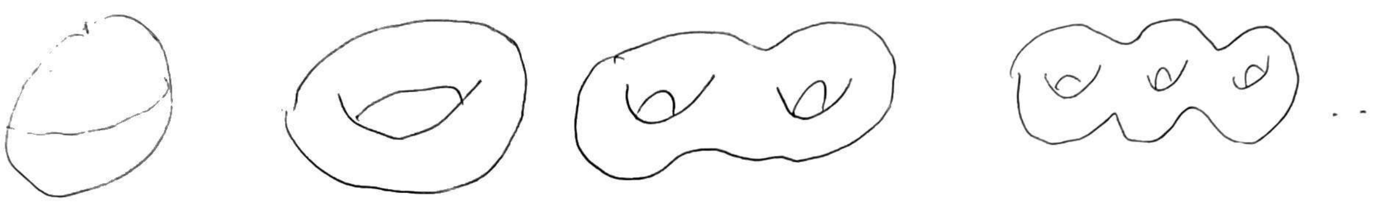
Therefore, on any topological sphere,

$$\int_S K\,dA = 4\pi$$

Classification of topological compact oriented surfaces without boundary.

g = # of holes $\qquad \chi(S) = 2-2g$

genus



$g=$	0	1	2	3	...
$\chi(S)$	2	0	-2	-4	...

Cor Suppose $S \subseteq \mathbb{R}^3$ is a compact oriented surface without boundary s.t. $K \geq 0$ and nonconstant, then S is homeomorphic to a sphere.

If $K \equiv 0$, then $S \cong T^2$ a torus.

Pf $\int_S K\,dA \geq 0$ with equality iff $K \equiv 0$

if $K>0$ $\qquad = \chi(S)$

$\chi(S) > 0 \Rightarrow S \cong S^2$

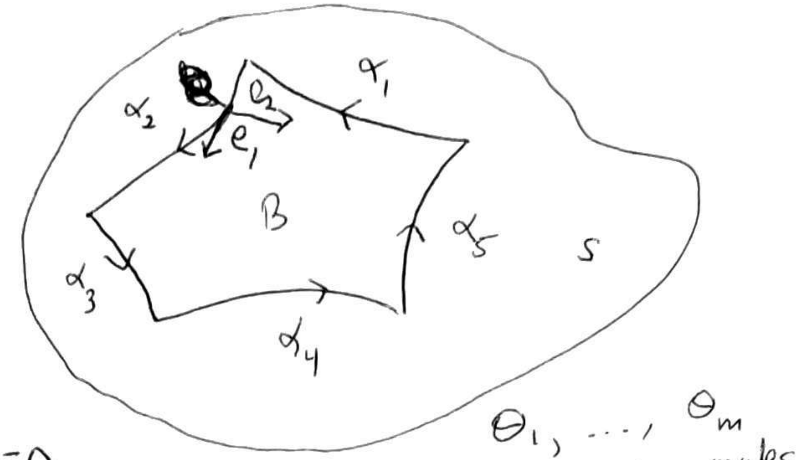
$\chi(S) = 0 \Rightarrow S \cong T^2$

Proof ~~S~~ Sketch of Gauss - Bonnet III

$\alpha : I \longrightarrow f(U) \subseteq S$

simple closed piecewise smooth curve with corners $t_1, \dots, t_m$

$f : U \longrightarrow S$ a chart



Pick f so that

$$g = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$$ is diagonal

$\theta_1, \dots, \theta_m$ exterior angles

$Y_i := \dfrac{\partial f}{\partial u^i}$ $\quad \langle Y_1, Y_2 \rangle = 0$

$X_i = Y_i / \| Y_i \|$ so $\{X_1, X_2, \nu\}$ is an orthonormal basis for $\mathbb{R}^3$

Suppose α par by arc length so

$\alpha' =: e_1$ normal let e_2 the unit normal to e_1 which is tangent to S

So $\{e_1, e_2\}$ ON-basis for $T_p S$

Choose so that orientation of $\{e_1, e_2\}$ & $\{X_1, X_2\}$ are the same

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

φ = angle that e_1 makes with X_1

Thm on turning tangents:

$$\int_\alpha \frac{d\varphi}{ds}\, ds = 2\pi - \sum_{i=1}^{m} \theta_i$$

for α a positively oriented simple closed curve with exterior angles θ_i.

$$\langle e_1, X_1 \rangle = \cos\varphi \qquad \frac{d}{ds}\langle e_1, X_1\rangle = -\sin\varphi \frac{d\varphi}{ds}$$

$$\frac{d\varphi}{ds} = \frac{-1}{\sin\varphi}\frac{d}{ds}\langle e_1, X_1\rangle = \frac{-1}{\sin\varphi}\left(\langle \nabla_{e_1} e_1, X_1 \rangle + \langle e_1, \nabla_{e_1} X_1 \rangle \right)$$

Plug in formular relating the two bases

$$= \frac{-1}{\sin\varphi}\Big(\langle \nabla_{e_1} e_1, e_1 \rangle \cos\varphi - \langle \nabla_{e_1} e_1, e_2 \rangle \sin\varphi$$

$$+ \langle X_1, \nabla_{e_1} X_1 \rangle \cos\varphi + \langle X_2, \nabla_{e_1} X_1 \rangle \sin\varphi \Big)$$

$\nabla_{e_1} e_1 =$ tangent to S component of $D_{e_1} e_1$

$$D_{e_1} e_1 = K_n \nu + K_g e_2$$

$$\langle \nabla_{e_1} e_1, e_1 \rangle = 0$$

$$\nabla_{e_1} e_1 = K_g e_2$$

$$\langle \nabla_{e_1} e_1, e_2 \rangle = K_g$$

$\langle X_1, X_1 \rangle$ constant so

$$0 = \nabla_{e_1} \langle X_1, X_1 \rangle = 2 \langle \nabla_{e_1} X_1, X_1 \rangle = 0$$

$\langle X_2, X_1 \rangle \equiv 0$

$$0 = \nabla_{e_1} \langle X_2, X_1 \rangle = \langle \nabla_{e_1} X_2, X_1 \rangle + \langle X_2, \nabla_{e_1} X_1 \rangle$$

$\Rightarrow$

$$\frac{d\varphi}{ds} = K_g - \langle \nabla_{e_1} X_1, X_2 \rangle$$

$$= K_g + \langle X_1, \nabla_{e_1} X_2 \rangle$$

$$e_1 = \alpha' = \frac{du^i}{ds} \frac{\partial f}{\partial u^i} = \frac{du^i}{ds} Y_i$$

$$\alpha = f(u^1(s), u^2(s))$$

$$\nabla_{e_1} X_2 = \frac{du^i}{ds} \nabla_{Y_i} X_2$$

$$\langle X_1, \nabla_{e_1} X_2 \rangle = \frac{du^1}{ds} \langle X_1, \nabla_{Y_1} X_2 \rangle + \frac{du^2}{ds} \langle X_1, \nabla_{Y_2} X_2 \rangle$$

$$2\pi - \sum \theta_i = \int_I \frac{d\varphi}{ds} ds = \int_I K_g \, ds + \int_I \left(\frac{du^1}{ds} \langle X_1, \nabla_{Y_1} X_2 \rangle + \frac{du^2}{ds} \langle X_1, \nabla_{Y_2} X_2 \rangle \right) ds$$

$$\int_I \left(\langle \nabla_{Y_1} X_2, X_1 \rangle \frac{du^1}{ds} + \langle \nabla_{Y_2} X_2, X_1 \rangle \frac{du^2}{ds} \right) ds \;=\!=$$

Green's Theorem

$$\int_B \left(\frac{\partial}{\partial u^1} \langle \nabla_{Y_2} X_2, X_1 \rangle - \frac{\partial}{\partial u^2} \langle \nabla_{Y_1} X_2, X_1 \rangle \right) du^1 du^2$$

$$\int_B \Big(\langle \nabla_{Y_1} \nabla_{Y_2} X_2, X_1 \rangle + \langle \nabla_{Y_2} X_2, \nabla_{Y_1} X_1 \rangle - \langle \nabla_{Y_2} \nabla_{Y_1} X_2, X_1 \rangle - \langle \nabla_{Y_1} X_2, \nabla_{Y_2} X_1 \rangle \Big) du^1 du^2$$

$$= \int_B \langle \overbrace{\nabla_{Y_1} \nabla_{Y_2} X_2 - \nabla_{Y_2} \nabla_{Y_1} X_2}^{R(Y_1, Y_2) X_2}, X_1 \rangle \, du^1 du^2$$

since $[Y_1, Y_2] = 0$ as Y_i are coordinate vector fields

$$+ \int_B \overbrace{\cancel{\Big(\langle \nabla_{Y_2} X_2, \nabla_{Y_1} X_1 \rangle - \langle \nabla_{Y_1} X_2, \nabla_{Y_2} X_1 \rangle \Big)}}^{0} du^1 du^2$$

$$= \int_B \langle R(Y_1, Y_2) X_2, X_1 \rangle \, du^1 du^2$$

$Y_1 = \sqrt{E}\, X_1$

$Y_2 = \sqrt{G}\, X_2$

$\sqrt{\det g}\, du^1 du^2 = dA$

$R(Y_1, Y_2) X_2 = \sqrt{EG}\, R(X_1, X_2) X_2$

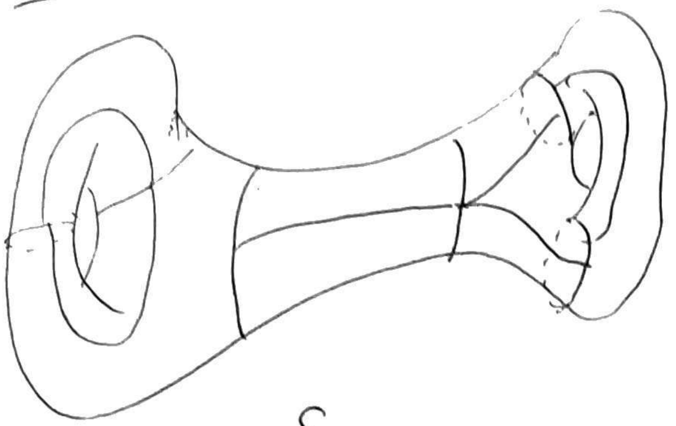
$$= \int_B \langle R(X_1, X_2) X_2, X_1 \rangle \underbrace{\sqrt{EG}\, du^1 du^2}_{\text{''}}$$

$\langle R(X_1, X_2) X_2, X_1 \rangle = K$

$$= \int_B K \, dA$$

$$2\pi - \sum \theta_i = \int_I \kappa_g \, ds + \int_B K \, dA \qquad \blacksquare$$

Proof sketch of global Gauss-Bonnet



S

Cut up S into finitely many polyhedral regions $\{M_i\}_{i=1}^m$ each contained in a chart

let θ_{ij} exterior angles of M_i

let ∂M_i = simple closed curve bounding M_i

$$\int_S K\,dA = \sum \int_{M_i} K \, dA$$

<u>GB III</u>

$$\int_{M_i} K\,dA + \int_{\partial M_i} \kappa_g \, ds = 2\pi - \sum_j \theta_{ij}$$

$$\sum_i \int_{\partial M_i} \kappa_g \, ds = 0$$ since the contributions cancel out

$$\int_S K\,dA = 2\pi m - \sum_{i,j} \overset{\text{exterior angles}}{\theta_{ij}} = 2\pi m - \sum_{i,j} (\pi - \overset{\text{interior angles}}{\varphi_{ij}})$$

using $\sum$ interior angle $= 2\pi$ at a vertex

$$= 2\pi (m - \#\text{edges} + \#\text{vertices})$$

$$= 2\pi (V - E + F) = 2\pi \chi(S) \qquad \blacksquare$$