**Rmk** Consider the $3^{rd}$ order Taylor expansion

$$c(s) = c(0) + s c'(0) + \frac{s^2}{2} c''(0) + \frac{s^3}{6} c'''(0) + o(s^3)$$

This can be rewritten as

$$c(s) = c(0) + \alpha(s) e_1(0) + \beta(s) e_2(0) + \gamma(s) e_3(0)$$

Using the Frenet eqns. (Hint: write $c'$, $c''$, & $c'''$ as functions of $e_i$)

## Osculating sphere

Let $c$ be a Frenet curve in $\mathbb{R}^3$ with $\tau(s_0) \neq 0$. Then there exits a unique sphere centered at

$$z(s_0) = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\tau(s_0)\kappa^2(s_0)} e_3(s_0)$$

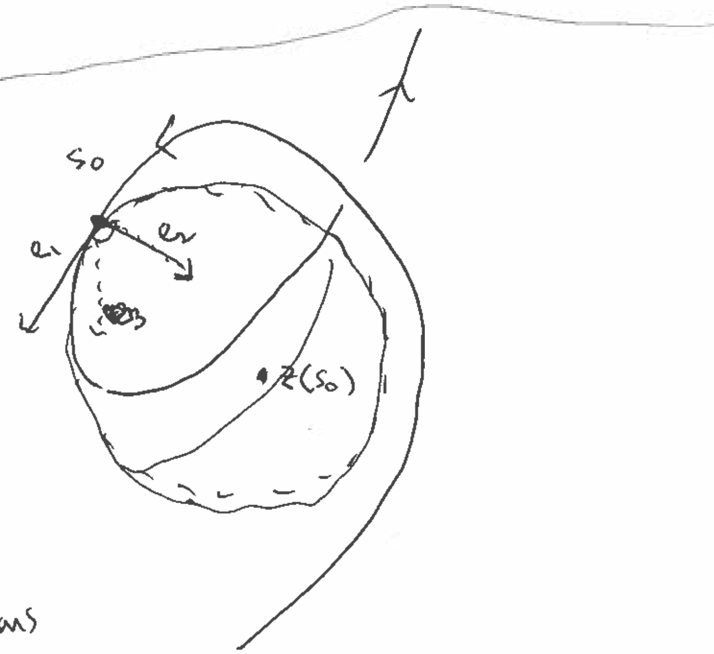
which is tangent to $c$ at $c(s_0)$ to order 3.

Write

$$z(s_0) = c(s_0) + \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0)$$

& consider the function

$$r(s) = \langle z(s_0) - c(s), z(s_0) - c(s) \rangle$$




The sphere we're looking for is given by $r(s) = \text{Constant}$ & being tangent to order 3 means

$$\boxed{r'(s_0) = r''(s_0) = r'''(s_0) = 0}$$

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want

$$r'(s_0) = \frac{d}{ds}\langle z(s_0)-c(s), z(s_0)-c(s)\rangle \Big|_{s=s_0}$$

$$= -2\langle z(s_0)-c(s), c'(s)\rangle \Big|_{s=s_0} = 0$$

$$\langle \alpha e_1(s_0)+\beta e_2(s_0)+\gamma e_3(s_0), e_1(s_0)\rangle = \boxed{\alpha = 0}$$

$$r''(s_0) = 2\langle c'(s), c'(s)\rangle - 2\langle z(s_0)-c(s), c''(s)\rangle \Big|_{s=s_0} = 0$$

$$1 - \langle \alpha e_1(s_0)+\beta e_2(s_0)+\gamma e_3(s_0), K e_2(s_0)\rangle = 0$$

$$1 - K\beta = 0 \qquad \boxed{\beta = \frac{1}{K}}$$

$$r'''(s_0) = 2\frac{d}{ds}\underbrace{\cancel{\langle c'(s), c'(s)\rangle}}_{0} + 2\underbrace{\cancel{\langle c'(s), c''(s)\rangle}}_{0} - 2\langle z(s_0)-c(s), c'''(s)\rangle \Big|_{s=s_0} = 0$$

$$\langle \alpha e_1(s_0)+\beta e_2(s_0)+\gamma e_3(s_0), K'(s_0)e_2(s_0) - K^2(s_0)e_1(s_0) + K(s_0)\tau(s_0)e_3(s_0)\rangle = 0$$

$$\boxed{\begin{gathered} C'''(s) = \\ K'e_2 + K(-Ke_1 + \tau e_3) \\ \text{by Frenet} \end{gathered}}$$

$$K'(s_0)\beta - \underbrace{\cancel{K^2(s_0)\alpha}}_{0} + K(s_0)\tau(s_0)\gamma = 0$$

$$\frac{K'(s_0)}{K(s_0)} + K(s_0)\tau(s_0)\gamma = 0$$

$$\boxed{\frac{-K'(s_0)}{K^2(s_0)\tau(s_0)} = \gamma}$$

So the sphere centered at

$$\boxed{z(s_0) = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) + \frac{-\kappa'(s_0)}{\kappa^2(s_0)\,\tau(s_0)} e_3(s_0)}$$

passing through $c(s_0)$ works, & its unique b/c we solved uniquely for the coefficients.

As $s$ varies, the center of the osculating sphere varies as ~~the~~ ~~[illegible]~~

$$z(s) = c + \frac{e_2}{\kappa} - \frac{\kappa'}{\kappa^2\tau} e_3$$

with radius given by $\| z(s) - c(s) \|$.

Thm (Spherical Curves)

let $c$ be a Frenet curve in $\mathbb{R}^3$ which is of class $C^4$ & suppose $\tau \neq 0$. Then, $c$ lies on a sphere iff

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\kappa'}{\tau \kappa^2}\right)$$

Proof $c$ lies on a sphere $\iff$ the osculating sphere is constant

$\Longleftrightarrow \quad z'(s) = 0 \quad \& \quad r'(s) = 0$

where $z(s)$ = center of the osculating sphere

$r(s) = \langle z(s) - c(s), z(s) - c(s) \rangle$ = radius$^2$

$$z'(s) = c' - \frac{\kappa'}{\kappa^2} e_2 + \frac{e_2'}{\kappa} - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right) e_3 - \frac{\kappa'}{\tau \kappa^2} e_3'$$

$$= e_1 - \frac{\kappa'}{\kappa^2} e_2 - \frac{\kappa e_1}{\kappa} + \frac{\tau e_3}{\kappa} - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right) e_3 + \frac{\kappa'}{\tau \kappa^2} \tau e_2$$

$$= \cancel{e_1} - \frac{\kappa' e_2}{\kappa^2} - \cancel{e_1} + \frac{\tau}{\kappa} e_3 - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right) e_3 + \cancel{\frac{\kappa' e_2}{\tau \kappa^2}}$$

$$= \left[\frac{\tau}{\kappa} - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right)\right] e_3 = 0 \Longleftrightarrow \boxed{\frac{\tau}{\kappa} - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right) = 0}$$

Thus, suppose $c$ lies on a sphere. Then $z'(s) \equiv 0$ so $\boxed{\frac{\tau}{\kappa} - \frac{d}{ds}\left(\frac{\kappa'}{\kappa^2 \tau}\right) = 0}$

Conversely suppose this equ holds, then $z'(s) = 0$. Now consider $r'(s) = \frac{d}{ds}\langle z(s) - c(s), z(s) - c(s) \rangle$

Using $z'(s) = 0$, we compute

$$r'(s) = -2 \langle z(s) - c(s), c'(s) \rangle$$
$$= -2 \left\langle \frac{1}{\kappa} e_2 - \frac{\kappa'}{\kappa^2 \tau} e_3, e_1(s) \right\rangle = 0$$

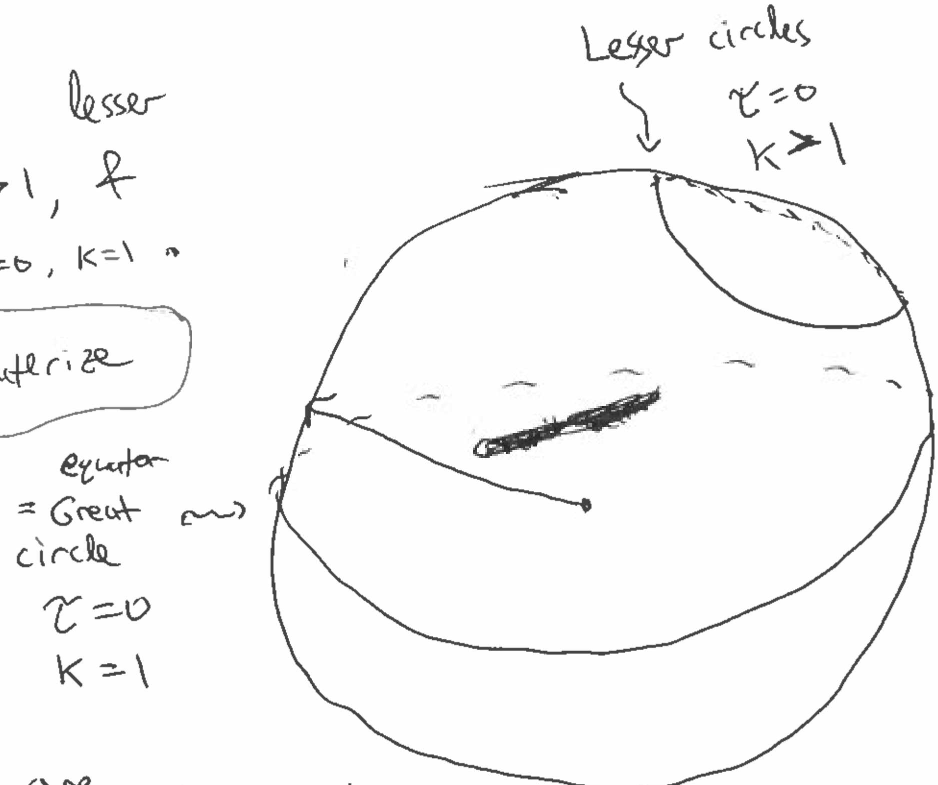
So the center & radius are constant & $c$ lies on a sphere $\square$

# Curves on a sphere

Consider $S^2 \subseteq \mathbb{R}^3$ the <u>unit sphere</u>

The sphere contains lesser circles with $\tau = 0$, $\kappa > 1$, & Great circles with $\tau = 0$, $\kappa = 1$.

How do we characterize these circles?




$c, c', c \times c'$ are an orthonormal basis

$$c'' = \langle c'', c \rangle c + \langle c'', c' \rangle c' + \langle c'', c \times c' \rangle c \times c'$$

Compute that $\langle c'', c' \rangle = 0$, $\langle c'', c \rangle = -\langle c', c' \rangle = -1$

$$c'' = -c + \underbrace{\langle c \times c', c'' \rangle}_{= \det(c, c', c'') =: J} c \times c' = \boxed{-c + J\, c \times c'}$$

$$\kappa^2 = \|c''\|^2 = 1 + J^2 \qquad \boxed{\kappa = \sqrt{1 + J^2}}$$

$$e_1 = c' \qquad e_2 = \frac{c''}{\kappa} \qquad e_3 = e_1 \times e_2 = \frac{c' \times c''}{\kappa}$$

$$\tau = -\langle e_3', e_2\rangle = -\left\langle \frac{d}{ds}\frac{c'\times c''}{\kappa}, \frac{c''}{\kappa}\right\rangle$$

Frenet

$$= -\frac{1}{\kappa^2}\Big\langle \underbrace{\frac{d}{ds}\, c'\times c''}_{=\, c'\times c''' \text{ by product rule}}, c''\Big\rangle + \frac{\kappa'}{\kappa^3}\cancel{\underbrace{\langle c'\times c'', c''\rangle}_{=\det(c', c'', c'')}}^{\,0}$$

$$= \frac{-1}{\kappa^2}\langle c'\times c''', -c + J\, c\times c'\rangle$$

$\langle c''', c\rangle = -\langle c'', c'\rangle = 0$ so $c'''$ & $c$ are $\perp$

b/c $\langle c'', c\rangle = -1$ is constant

$\Rightarrow$ $c'\times c''' \perp c'\times c$

$\Rightarrow$ $\langle c'\times c''', c'\times c\rangle = 0$

$$= \frac{+1}{\kappa^2}\langle c'\times c''', c\rangle = \frac{1}{\kappa^2}\det(c, c', c''')$$

Note $\underbrace{\frac{d}{ds}\det(c, c', c'')}_{=\, J'} = \underbrace{\det(\cancel{c'}, c', c'')}_{0} + \underbrace{\det(c, \cancel{c''}, c'')}_{0} + \det(c, c', c''')$

so $\boxed{\tau = \frac{J'}{\kappa^2}}$

Thus, $\kappa = \sqrt{1+J^2}$

$\tau = \frac{J'}{1+J^2}$

$J$ = geodesic curvature (we'll learn about this later)

It measures the curvature of $c$ relative to that of the sphere.

$J = 0 \Leftrightarrow c$ is a great circle $\Leftrightarrow$ $c$ is "as flat as possible on the sphere"

$J$ = constant $\Leftrightarrow$ $c$ is a lesser circle $\Leftrightarrow$ ... sphe...