Curves with constant slope

Thm for a Frenet curve in $\mathbb{R}^3$, TFAE

1) there is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ s.t. $\langle e_1, v\rangle$ = constant
2) " " " s.t. $\langle e_2, v\rangle = 0$
3) " " " s.t. $\langle e_3, v\rangle$ = constant
4) $\frac{\tau}{\kappa}$ is constant

Pf if $\tau = 0$ then c is planar contained on $e_3^\perp$ & so $v = e_3$ works.

if $\tau \neq 0$,

1) $\langle e_1, v\rangle$ = constant $\Leftrightarrow$ $0 = \frac{d}{ds}\langle e_1, v\rangle = \langle e_1', v\rangle = \kappa\langle e_2, v\rangle = 0$

$\left(\text{but } \kappa \neq 0\right)$ $\Leftrightarrow$ $\langle e_2, v\rangle = 0$

similarly ~~[illegible]~~ $\langle e_3, v\rangle$ = constant $\Leftrightarrow$

$$\frac{d}{ds}\langle e_3, v\rangle = \langle e_3', v\rangle = -\tau\langle e_2, v\rangle = 0$$

so $\langle e_2, v\rangle = 0$

thus 1, 2, 3 equivalent wherever $\tau \neq 0$

~~[illegible]~~ if 1, 2, 3, write $v = \alpha e$ the constant then $0 = \alpha e_1' + \beta e_3' = \alpha \kappa e_2 - \beta \tau e_2 = 0$

So $\dfrac{\alpha}{\beta} = \dfrac{\tau}{\kappa}$ is constant

$\boxed{\beta \neq 0, \text{ why?}}$

Conversely if $\dfrac{\tau}{\kappa}$ is constant,

then $N = \dfrac{\tau}{\kappa} e_1 + e_3$ is constant

& this N works.

<u>Special case</u> what if both τ & κ are constant?

Consider the helix

$$c(t) = (a\cos(\alpha t), a\sin(\alpha t), bt)$$

with $1 = \alpha^2 a^2 + b^2$ (arc length)

then $\kappa^2 = \alpha^4 a^2$ $\quad \tau^2 = \alpha^2 b^2$ are constant

Conversely, fix τ, κ, you can solve for α, a & b to find a helix.

Uniqueness of solutions to ODE ~~given~~ means this is the only solution.

A Frenet curve in $\mathbb{R}^3$ has constant $\tau, \kappa \iff$ its a helix

Thm (Frenet eqns) let c be a Frenet curve in $\mathbb{R}^n$ with Frenet frame $e_1, \ldots, e_n$. There exist function $\kappa_1, \ldots, \kappa_{n-1}$ s.t.

1) $\kappa_1, \ldots, \kappa_{n-2} > 0$

2) κ_i is $(n-1-i)$-times cont differentiable

3)

$$\frac{d}{ds}\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & & & 0 \\ -\kappa_1 & 0 & \kappa_2 & & \\ & -\kappa_2 & \ddots & \ddots & \\ & & \ddots & 0 & \kappa_{n-1} \\ 0 & & & -\kappa_{n-1} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

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e.g for $n=3$, we have $\kappa = \kappa_1 > 0$ the curvature, $\tau = \kappa_2$ the torsion.

Rmk 1 if c is C^∞ then so are the κ_i

Rmk 2 κ_{n-1} is called the torsion

$$e_n' = -\kappa_{n-1} e_{n-1}$$ so $\kappa_{n-1} = 0 \iff e_n$ is constant $\iff$ c lies on the hyperplane $e_n^\perp$ perpendicular

Frenet Curves in $\mathbb{R}^n$

Def ① A regular Curve ~~curve~~ c in $\mathbb{R}^n$ is **Frenet**

if at every point, the vectors

$$c', \ldots, c^{(n-1)} \quad \text{(first } (n-1) \text{ derivatives)}$$

are linearly independent.

② Given a Frenet curve c, the **Frenet frame** $e_1, \ldots, e_n$ is the unique basis (depending on s) s.t.

1) $e_1, \ldots, e_n$ orthonormal & positively oriented

2) for each $k = 1, \ldots, (n-1)$,
$$\mathrm{Span}(e_1, \ldots, e_k) = \mathrm{Span}(c', \ldots, c^{(k)})$$

3) $\langle c^{(k)}, e_k \rangle > 0$ for each $k = 1, \ldots, (n-1)$.

Rmk given a frenet c,

$$c', \ldots, c^{(n-1)} \overset{\text{gram-schmidt}}{\rightsquigarrow} e_1, \ldots, e_{n-1}$$

e_n is the unique positively oriented unit vector in $\mathrm{Span}(e_1, \ldots, e_{n-1})^{\perp}$.

Proof of theorem

Write

$$e_i' = \sum_{j=1}^{n} \langle e_i', e_j \rangle e_j$$

for $i \leq n-1$, $\quad e_i \in \operatorname{Span}(c', \ldots, c^{(i)})$ so

$$e_i' \in \operatorname{Span}(c', \ldots, c^{(i+1)}) = \operatorname{Span}(e_1, \ldots, e_{i+1})$$

therefor, $\quad \langle e_i', e_j \rangle = 0 \quad$ for $\quad j = i+2, i+3, \ldots, n$

Set $\quad K_i = \langle e_i', e_{i+1} \rangle$

$\underline{i=1}$ $\quad e_1' \in \operatorname{Span}(e_1, e_2) \qquad$ but $\; 2\langle e_1', e_1 \rangle = \frac{d}{ds}\langle e_1, e_1 \rangle = 0$

so $\quad e_1' = K_1 e_2$

$\underline{i=2}$ $\quad e_2' \in \operatorname{Span}(e_1, e_2, e_3) \qquad \langle e_2', e_2 \rangle = 0$

$$e_2' = \langle e_2', e_1 \rangle e_1 + \langle e_2', e_3 \rangle e_3 = -K_1 e_1 + K_2 e_3$$

$$\langle e_2', e_1 \rangle + \langle e_2, e_1' \rangle = \frac{d}{ds}\langle e_2, e_1 \rangle = 0$$

$$\langle e_2', e_1 \rangle = -\langle e_1', e_2 \rangle = -K_1$$

In general, note that

$$\boxed{\begin{gathered}\langle e_i', e_j \rangle = -\langle e_i, e_j' \rangle \\ \langle e_i', e_i \rangle = 0\end{gathered}}$$

$$0 = \frac{d}{ds}\langle e_i, e_j \rangle = \langle e_i', e_j \rangle + \langle e_i, e_j' \rangle$$

thus the matrix is skew symmetric and all entries are zero except

$$\langle e_i', e_{i+1} \rangle = k_i \quad \& \quad \langle e_i', e_{i-1} \rangle = -k_{i-1}$$

k_i is C^{n-1-i} by construction

Finally, if $i \leq n-2$,

$k_i = \langle e_i', e_{i+1} \rangle$ has the same sign as $\langle c^{(i+1)}, e_{i+1} \rangle > 0$. ↯

<u>Notation</u> $\{k_i\}_{i=1}^{n-1}$ = Frenet curvatures of the matrix is denoted K

So the eq^n can be written as $F' = K F$.

$F = [e_1, \ldots, e_n]^T$

<u>Rmk</u> Note that the Frenet curvatures & Frenet frame are invariant under

<u>Euclidian transformations</u> : $B : \mathbb{R}^n \to \mathbb{R}^n$ $\quad Bv = Av + b$

$$A^{-1} = A^T, \quad \det(A) = 1$$

That is, $B \circ c$ is a Frenet curve with frame $Ae_1, \ldots, Ae_n$, & curvatures k_i