Thm (Fundamental Thm of local theory of curves)

let $\kappa_1, \dots, \kappa_{n-1} : (a,b) \to \mathbb{R}$ be C^∞ functions with $\kappa_1, \dots, \kappa_{n-2} > 0$. Suppose for $s_0 \in (a,b)$, we fix a point $q_0 \in \mathbb{R}^n$ as well as an n-frame $e_1(s_0), \dots, e_n(s_0)$. Then there is a unique C^∞ Frenet curve $c : (a,b) \to \mathbb{R}^n$ par. by arc length, such that

1) $c(s_0) = q_0$,
2) $e_1(s_0), \dots, e_n(s_0)$ is the Frenet frame of c at s_0
3) $\kappa_1, \dots, \kappa_{n-1}$ are the Frenet curvatures.

More generally, if κ_i is C^{n-1-i} for each i, then there is a unique such c in class C^n.

Proof idea

curvature matrix

Write the Frenet eqns as $F' = KF$
where $F = [e_1, \dots, e_n]^T$ is a matrix of unknowns
initial conditions: $F(s_0) = [e_1(s_0), \dots, e_n(s_0)]$

General theory of differential equations
$\implies$ there is a unique solution $F(s)$ for $s \in (a,b)$ with $F(s_0) = [e_1(s_0), \dots, e_n(s_0)]$

Claim $F(s)$ is a positive orthonormal frame for each $s \in (a,b)$, that is, $F(s)$ is an orthogonal matrix

$$FF^T = Id$$
$$\& \ \det F = 1$$

Indeed, $\frac{d}{ds}(\underbrace{FF^T}_{G}) = F'F^T + F(F')^T$

$$= KFF^T + FF^TK^T$$

$$\frac{dG}{ds} = KG + GK^T$$ is a Diff eq for G

with initial value $G(s_0) = Id$,

but $K + K^T = 0$ so $G \equiv Id$ must be the unique solution, thus $FF^T = G \equiv Id$.

$\det(FF^T) = \det(Id)$

$\det(F)^2 = 1$

$\det(F) = \pm 1$

So by continuity,

$\det(F(s_0)) = 1 \Rightarrow \det(F) = 1$.

Now we wish to find a curve with this frame

$$c(s) = q_0 + \int_{s_0}^{s} e_1(t)\,dt$$

then $c'(s) = e_1(s)$

& the other e_i can be checked to be the Frenet frame.

Global Theory of Plane Curves

Def a regular curve $\gamma : [a,b] \to \mathbb{R}^n$ is **closed** if $\gamma(a)=\gamma(b)$, $\gamma'(a)=\gamma'(b)$, $\gamma''(a)=\gamma''(b)$, etc

(all derivatives agree up to n if its C^a)

γ a closed curve is **simple** if $\gamma|_{[a,b)}$ is injective.

Rmk 1) equivalent notions: a) $\gamma : S^1 \to \mathbb{R}^n$ where S^1 = circle
C^a & regular

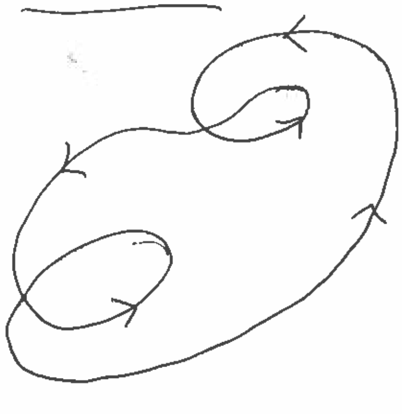
b) $\gamma : \mathbb{R} \to \mathbb{R}^n$ C^a & regular
with $\gamma(r+b-a) = \gamma(r)$

c) regular closed curve

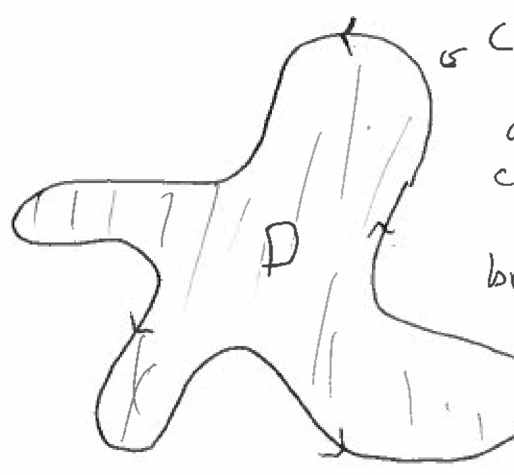
$$\boxed{C := \text{im}(\gamma)}$$

Thm (Jordan curve theorem)

Any simple close curve in $\mathbb{R}^2$ bounds a unique region, called the interior of C



a close curve



a simple closed curve bounding D

<u>Question</u> how are the area of D & the <u>length</u> of C related?

L = length(C) $\quad A$ = area(D)

C = circle

$L = 2\pi r \quad A = \pi r^2$

$L^2 = 4\pi A$

<u>Thm</u> (Isoperimetric inequality)

let C be a simple closed curve with length L & bounding a region of area A. Then

$$L^2 - 4\pi A \geq 0$$

<u>with equality</u> $\Leftrightarrow$ C is a circle.

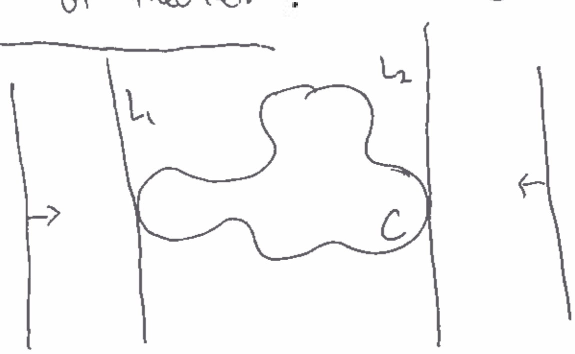
<u>Lemma</u> write a simple closed curve as $\alpha : [a,b] \to \mathbb{R}^2$. with $\alpha(t) = (x(t), y(t))$

then

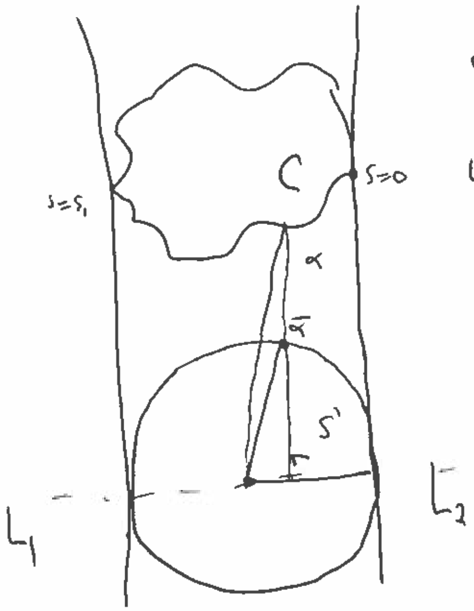
$$A = -\int_a^b y(t)\,x'(t)\,dt = \int_a^b x(t)\,y'(t) = \frac{1}{2}\int (xy' - yx')\,dt$$

<u>Pf</u> multivariable calc Green's theorem or

<u>Proof of theorem:</u> $C = \{\alpha(s) = (x(s), y(s))\}$



Pick a circle S^1 disjoint from C & tangent to L_1 & L_2



$\mathrm{dist}(L_1, L_2) = 2r$

write s' as

$\bar{\alpha}(s) = (x(s), \bar{y}(s))$

↑ same x value

$$A = \int_0^\ell x y' \, ds \qquad \bar{A} = \pi r^2 = -\int_0^\ell \bar{y} x' \, ds$$

$$A + \pi r^2 = \int_0^\ell (xy' - \bar{y} x') \, ds \le \int_0^\ell \sqrt{\underbrace{(xy' - \bar{y}x')^2}_{//}} \, ds \le \int_0^\ell \sqrt{\underbrace{(x^2 + \bar{y}^2)}_{= r^2}\underbrace{(x'^2 + y'^2)}_{1 = \|\alpha\|}} \, ds = \ell r$$

$\left\langle \begin{bmatrix} x \\ \bar{y} \end{bmatrix}, \begin{bmatrix} y' \\ -x' \end{bmatrix} \right\rangle \le (x^2 + \bar{y}^2)^{1/2} (x'^2 + y'^2)^{1/2}$ by Cauchy-Schwarz

AM - GM

$(AB)^{1/2} \le \dfrac{A+B}{2}$ with equality $\iff A = B$

By AM - GM,

$$\sqrt{\pi r^2 A} \le \frac{1}{2}(A + \pi r^2) \le \frac{1}{2} \ell r$$

$$4\pi A r^2 \le \ell^2 r^2 \implies \boxed{4\pi A \le \ell^2}$$

Suppose $\ell^2 = 4\pi A$, then by AM-GM, $A = \pi r^2$, so $\ell = 2\pi r$

So we have equality

$$A + \pi r^2 = \int_0^\ell (\text{—}) \, ds = \int_0^\ell (\text{——}) \, ds = \ell r \quad \text{above}$$

so by Cauchy-Schwarz, $(x, \bar{y}) = \lambda (y', -x')$ for some λ

$$\lambda = \frac{x}{y'} = \frac{\bar{y}}{x'} = \frac{\|(x,\bar{y})\|}{\|(y', x')\|} = \frac{\sqrt{x^2+\bar{y}^2}}{\sqrt{(y')^2+(x')^2}} = \pm r$$

$x = \pm r y'$ now repeat the argument with different orientation to obtain $y = \pm r x'$

then $x^2 + y^2 = r^2$ so $\alpha(s) = (x(s), y(s))$ lies on a circle

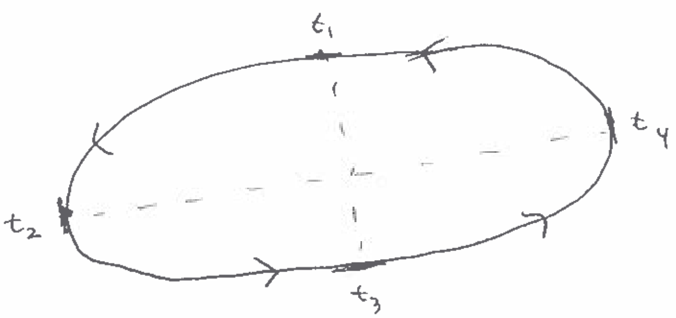
Since C is simple closed, it must be equal to the circle. ∎

The Four Vertex theorem +

Def A vertex of a regular plane curve $C : \{\alpha : [a,b] \to \mathbb{R}^2\}$ is a point where $K(t)$

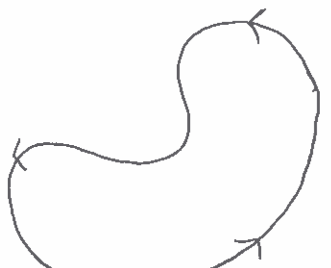
Ex a) if C has constant curvature, then every point is a vertex.

b)



an ellipse which isn't a circle has 4 vertices

c)



more than 4 vertices

not convex!

Def 1) a region D is convex if for all $a, b \in D$, the line segment from a to b is contained in D.

2) a simple closed curve is convex if the region D it bounds is convex.

Thm TFAE (Kü 2.31)

1) C is convex 2) each line intesects C in either a segment 2 points, or 1 pt

3) for each point $p \in C$, C lies completely to one side of T_pC

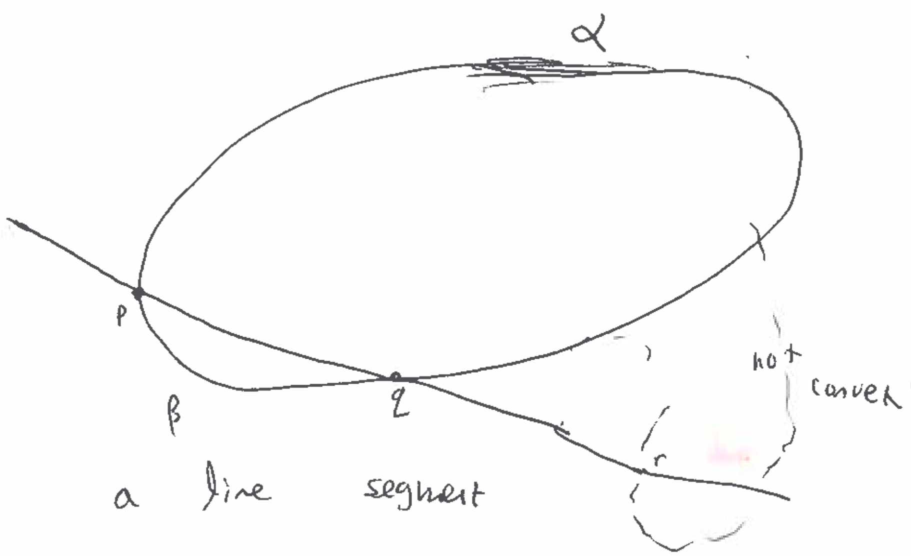
4) the curvature does not change sign

The Four vertex theorem

A simple closed convex curve has at least 4 vertices.

Pf Write C as $\gamma : [0, \ell] \to \mathbb{R}^2$ arc length par. Since κ is a continuous function, it achieves a maximum & minimum on the closed interval I. Thus, it has at least 2 vertices $p, q \in C$, say the max & min respectively

Now consider the line L through p, q.



By convexity, either

$L \cap C = \{p, q\}$ or a line segment

Case 1 $\quad L \cap C$ is a line segment,

then $\kappa = 0$ at p & q but these are the max & min so $\kappa \equiv 0$

$\Longrightarrow$ C is a line segment

Case 2 $\quad L \cap C = \{p, q\}$ & $\kappa' \neq 0$

then L divides C into two arcs, α, β

If there are not other vertices, then κ' has constant sign on α & β. Suppose wlog that L is the x-axis. Then

$y\kappa'$ has the same sign everywhere

write $x' = \sin\theta$, $y' = \cos\theta$

Frenet eqn $\Longrightarrow$ $\kappa y' = x''$

$$\int_0^\ell y\kappa' \, ds = \cancel{y\kappa}\Big|_0^\ell - \int_0^\ell \kappa y' \, ds$$

$$= x'(0) - x'(\ell) = 0. \Longrightarrow \kappa' = 0, \text{ contradiction!}$$