Local Theory of Surfaces

Def A <u>parametrized surface element</u> is a continuously differentiable map $f: U \to \mathbb{R}^3$, $U \subseteq \mathbb{R}^2$. We call it <u>regular</u> if

$$df_p : \mathbb{R}^2 \to \mathbb{R}^3$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}\Bigg|_p$$

<u>has full rank</u> for each $p \in U \subseteq \mathbb{R}^2$ open.

$$\boxed{\begin{aligned} f(u,v) &= (f_1(u,v), f_2(u,v), f_3(u,v)) \\ f(u,v) &= (x(u,v), y(u,v), z(u,v)) \end{aligned}}$$

A <u>regular surface element</u> is an equivalence class of regular par. surface elements up to reparametrization:

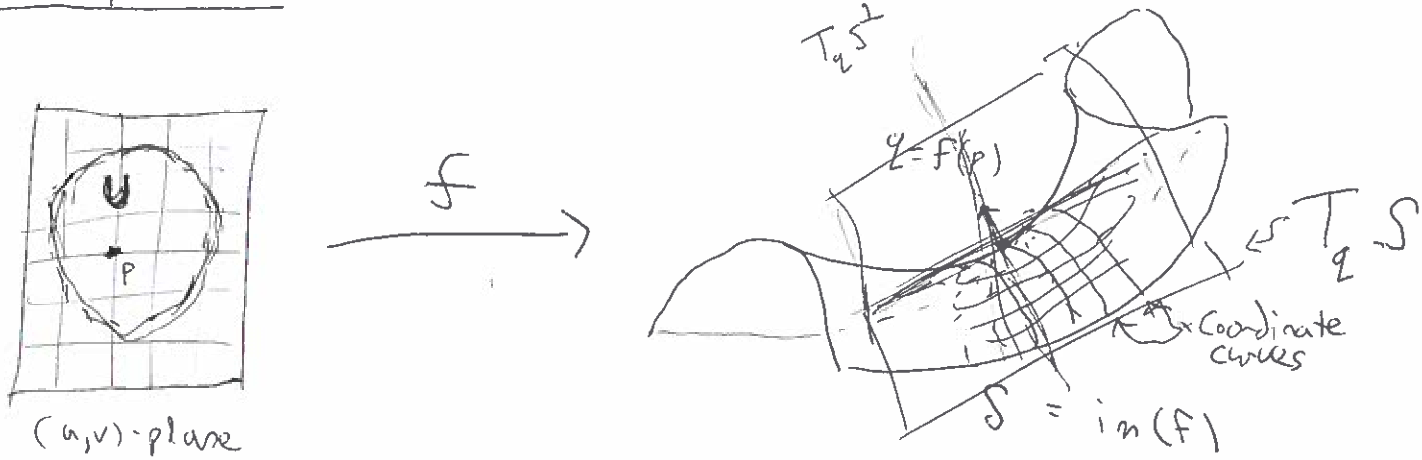
$$\begin{tikzcd} f: U \subseteq \mathbb{R}^2 \arrow[r] & \mathbb{R}^3 \\ f': U' \subseteq \mathbb{R}^2 \arrow[u, "\varphi"] \arrow[ur] & \end{tikzcd}$$

φ is a <u>diffeomorphism</u> (a C^1 function with C^1 inverse)

$$f(u,v) = f'(\varphi(u',v'))$$

$$f = f' \circ \varphi$$

Notation / Remark



$$\begin{tikzcd} T_pU = \mathbb{R}^2 \arrow[rr, "dF \text{ is injective}"] \arrow[dr] & & \mathbb{R}^3 = T_q\mathbb{R}^3 \\ & \mathrm{Im}(df) =: T_qS \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

$T_qS^\perp$ = perpendicular to T_qS in $T_q\mathbb{R}^3$ WRT euclidian inner product on $\mathbb{R}^3$

Note that $\mathrm{Im}(df) = \mathrm{Span}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$

So f is regular $\iff$ $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$ are linearly indepentent for all $u,v \in U$

Coordinate curves on S given by

$\gamma_u(t) = f(p + (t,0))$ or $\gamma_v(t) = f(p + (0,t))$

$\dot{\gamma}_u\Big|_{t=0} = \frac{\partial f}{\partial u}\Big|_p$ $\qquad$ $\dot{\gamma}_v(t)\Big|_{t=0} = \frac{\partial f}{\partial v}\Big|_p$

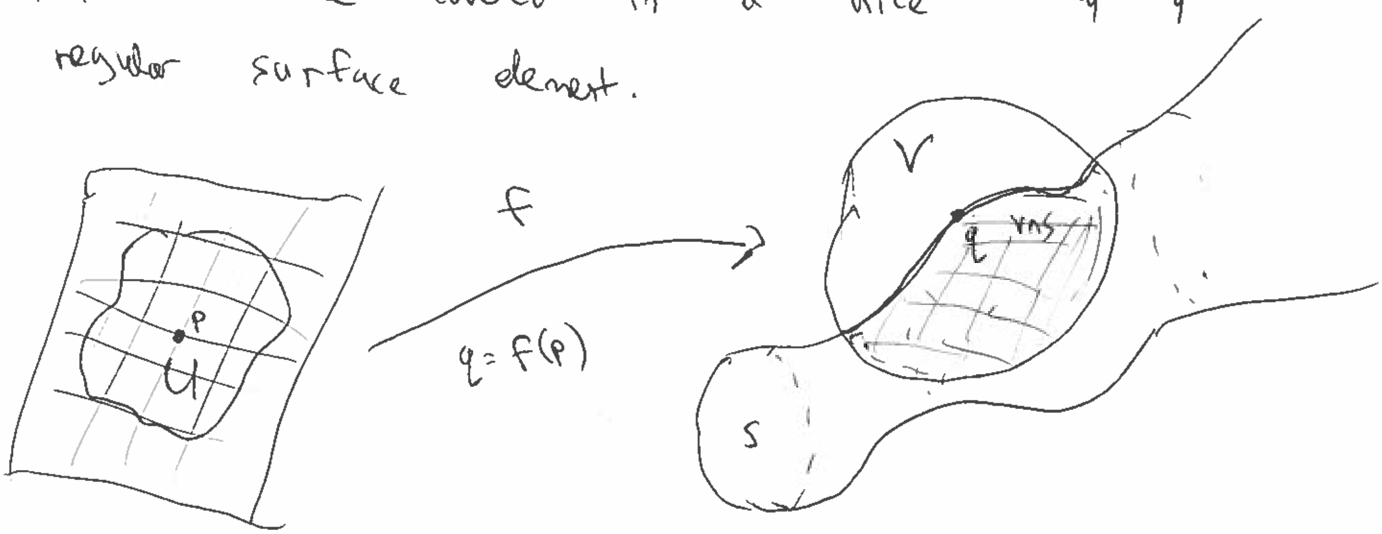
Def A __regular__ surface $S \subseteq \mathbb{R}^3$ is a subset such that for each $q \in S \subseteq \mathbb{R}^3$, there exists an open neighborhood $q \in V \subseteq \mathbb{R}^3$ & a regular parametrization

$$f : \underset{\mathbb{R}^2}{\overset{\cap\!\mid}{U}} \longrightarrow V \cap S \subseteq \mathbb{R}^3$$

such that

f is a bijection of U onto $V \cap S$ with continuous inverse. (Homeomorphism, but in fact by IFT, it's actually diffeomorphism)

That is, a regular surface is a subset $S \subseteq \mathbb{R}^3$ that can be covered in a nice way by regular surface element.



Note that the condition is independent of choice of regular parametrization:

$$\mathbb{R}^2 \supseteq U \xrightarrow{f} \overset{q \in}{V \cap S} \subseteq V$$

$$V'' \cap S$$

$$V'' = V \cap V'$$

$$\mathbb{R}^2 \supseteq U' \xrightarrow{f'} V' \cap S \ni q \subseteq V'$$

$$\begin{tikzcd} & f^{-1}(V''\cap S) \subseteq U \arrow[dl, "f"'] \\ V''\cap S & f'^{-1}(V''\cap S) \subseteq U' \arrow[l, "f'"] \arrow[u, "\varphi"] \end{tikzcd}$$

$$\varphi = f\big|_{V''\cap S}^{-1} \circ f'\big|_{f'^{-1}(V''\cap S)}$$

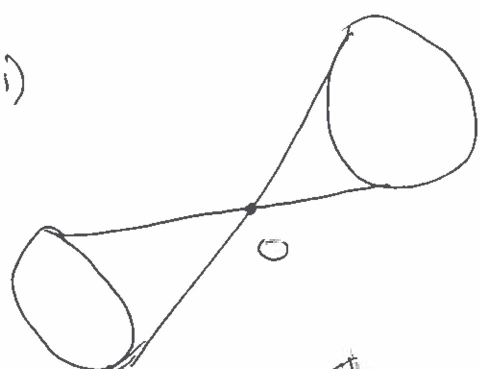
Gives an equivalence between $\left(f\big|_{f^{-1}(V''\cap S)}\right)$ & $\left(f'\big|_{f'^{-1}(V''\cap S)}\right)$

This definition works more generally for subsets of $\mathbb{R}^n$ parametrized by open sets of $\mathbb{R}^m$, $m < n$

Gives the notion of an <u>embedded</u> <u>manifold</u>: (submanifold)

So reg surfaces are 2-dim submanifolds of $\mathbb{R}^3$

<u>Ex</u> 1)



<u>Not</u> a submanifold

no regular parametrization at O

2)

regular surface element but not a submanifold

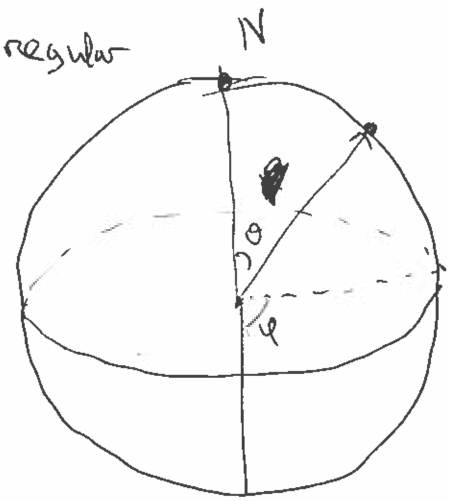
<u>Prop</u> Given a regular par. surface $f: U \longrightarrow \mathbb{R}^3$, there exists a small enough open set $V \subseteq U$ s.t. $f(V)$

Pf

$f: U \longrightarrow \mathbb{R}^3$, since df_p has maximal rank by assumption
$\quad\ \ \rotatebox{90}{$\in$}$
$\quad\ \ p$

by Inverse function theorem, there exists a $p \in V \subseteq U$ such that $f|_V : V \to f(V)$ has a C^1 inverse so $f(V)$ is a regular surface (embedded submanifold).

3) the sphere is a regular surface



$\subseteq \mathbb{R}^3$

$x^2 + y^2 + z^2 = 1$

many parametrizations

~~stereographic projection~~

e.g. consider

$\left(u, v, \sqrt{1-u^2-v^2}\right)$

works for $z > 0$,

$\left(u, v, -\sqrt{1-u^2-v^2}\right)$

works for $z < 0$

⋮

$\left(u, \sqrt{1-u^2-v^2}, v\right)$

$y > 0$

⋮

etc

Each hemisphere has a regular parametrization of the form on the right.

another parametrization:

$f(\varphi, \theta) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$

$\theta \in (0, \pi) \qquad \varphi \in (0, 2\pi)$

Ex Check that this is a regular parametrization. What is its image?