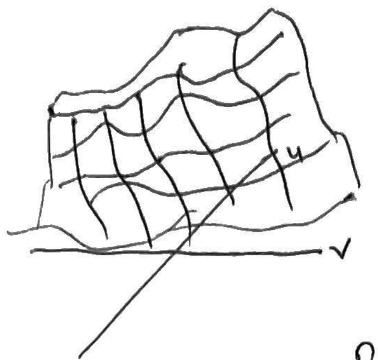
4) The graph of a function $h: \mathbb{R}^2 \to \mathbb{R}$
cont. differentiable

$f: \mathbb{R}^2 \to \mathbb{R}^3 \qquad f(u,v) = (u, v, h(u,v))$

$\dfrac{\partial f}{\partial u} = (1, 0, h_u)$

$\dfrac{\partial f}{\partial v} = (0, 1, h_v)$



$J(f) = \begin{bmatrix} 1 & 0 & h_u \\ 0 & 1 & h_v \end{bmatrix}$ is always full rank

by Inverse Fcn

the projection $Pr\big|_{Im(f)} : Im(f) \to \mathbb{R}^2$ gives us the inverse

---

Thm: Suppose S is a regular surface, & $p \in S$. Then there exist coordinates & an open nbhd $p \in V \subseteq S$ s.t. V is the graph of a cont. diff function $h: U \to \mathbb{R}^3$, $U \subseteq \mathbb{R}^2$

Pf: Suppose $f(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parametrization. By assumption, one of the 2×2 minors of Jf is nonzero, say

$$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_p = \det \left.\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}^T\right|_p \neq 0,$$

now project to the (x,y) plane and apply the Inverse Function theorem.

5) <u>Regular values</u>

Let $F : \underset{\underset{U}{\cup}}{\mathbb{R}^3} \longrightarrow \mathbb{R}$ (or even $\mathbb{R}^n \longrightarrow \mathbb{R}$)

A point $p \in U$ is a <u>critical point</u> if $JF_p = 0$. A point $q \in \mathbb{R}$ is a <u>regular value</u> if there are no critical points in $F^{-1}(q)$.

<u>Theorem</u> If q is a regular value, $F^{-1}(q) \subseteq \mathbb{R}^3$ $(\subseteq \mathbb{R}^n)$ is a regular (hyper) surface

<u>Pf</u>: Suppose wlog $F_z \neq 0$ at $p \in F^{-1}(q)$
Consider $G(x,y,z) = (x, y, F(x,y,z))$

<u>Claim</u> JG_p is invertible for each $p \in F^{-1}(q)$

$$JG_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x & F_y & F_z \end{bmatrix} \qquad \det(JG_p) = F_p(p) \neq 0$$

By inverse function theorem, $\exists$ a nbhd ~~...~~ $W = G(V) \ni G(p)$, $p \in V$ s.t. $G|_V$ is invertible, inverse $G^{-1} : W \to V$ is of the form

$$G^{-1}(a,b,c) = (a, b, g(a,b,c))$$

$$(a,b,c) = (x, y, F(x,y,z))$$

In particular, on the level set $F = q$, we have $c = q$ & ~~[illegible]~~ $h(u,v) = g(u,v,q)$ is a C^1 function & the graph of h is exactly the locus $F = q$. $\boxed{q}$

Rmk 1) diff functions on ~~[illegible]~~ regular hypersurfaces.

Given $S \subseteq \mathbb{R}^3$, ~~[illegible]~~ $V \subseteq \mathbb{R}^3$ open

~~[illegible]~~ a differentiable function on $S \cap V$ can be defined as a function of the form ~~[illegible]~~ $\varphi : S \cap V \longrightarrow \mathbb{R}$ s.t.

$\varphi \circ f$ is diff where

$f : \underset{\mathbb{R}^2}{\overset{\cap}{U}} \longrightarrow S \cap V$ is a chart. If S has C^∞ charts, then we can define φ being C^∞

2) ~~[illegible]~~ ~~[illegible]~~ a C^1 (or C^n or C^∞) curve on S can be defined as a map

$\alpha : [a,b] \longrightarrow S \subseteq \mathbb{R}^3$ which is C^1 (C^n or C^∞).

Suppose $(-\varepsilon, \varepsilon) \ni 0$, $\alpha(0) = p$ then $\dot{\alpha}(0) \in T_p S$

we can identify $T_p S = \left\{ \begin{array}{l} \text{Equivalence} \\ \text{classes of} \\ \text{curves } \alpha \\ \text{s.t. } \alpha \sim \beta \\ \Leftrightarrow \dot{\alpha}(0) = \dot{\beta}(0) \end{array} \right\}$

Given a vector $X \in T_pS$, & $f \in C^\infty(V)$ (9)

Sets of smooth functions (or diff if C^n/C^1)

We can define

$$X(f) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$$ ← directional derivative in the X direction

Where α is a curve with $\dot{\alpha}(0) = X$.

3 different descriptions of tangent vectors:

$$T_pS = df_q(\mathbb{R}^2)$$ Where $f: U \to V \cap S$, $q \in U$, $p \in V \cap S$ is a chart

$$= \left\{ [\dot{\alpha}(0)] \;\middle|\; \begin{array}{l} \alpha: (-\varepsilon, \varepsilon) \to S \\ \alpha(0) = p, \ \alpha \sim \beta \\ \text{if } \alpha'(0) = \beta'(0) \end{array} \right\}$$

$$= \left\{ X: C^\infty(V) \to \mathbb{R} \text{ of directional derivatives} \right\}$$

note $X(fg)$

$= fX(g) + X(f)g$

Product or Leibniz rule

Coordinate curves in $\mathbb{R}^2$ $\leadsto$ $\dot{\gamma}_u$, $\dot{\gamma}_v$

$\gamma_u = f(q + t(1,0))$

$\gamma_v = f(q + t(0,1))$

$\leadsto$ $X_u(g) = \frac{d}{dt} g(f(q+t(1,0)))\Big|_{t=0}$

$X_v(g) = \frac{d}{dt} g(f(q+t(0,1)))\Big|_{t=0}$

$\leadsto \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$

Note here also that we can consider all of these as functions of p as well

Def (1st fundamental form)

let $\langle \, , \rangle$ be the euclidian inner product on $\mathbb{R}^3$ (resp $\mathbb{R}^n$)

the first fundamental form

$$\mathrm{I} : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\mathrm{I}(X, Y) = \langle X, Y \rangle$$

If $f(u,v) = (x(u,v), y(u,v), g(u,v))$ is a parametrization, we can write it as follows:

$$(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathrm{I}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial u}\right) & \mathrm{I}\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) \\ \mathrm{I}\left(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}\right) & \mathrm{I}\left(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial v}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u} \right\rangle & \left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right\rangle \\ \left\langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u} \right\rangle & \left\langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial v} \right\rangle \end{pmatrix}$$

g = metric tensor

$$\mathrm{I}(X, Y) = (a, b)\, g \begin{pmatrix} c \\ d \end{pmatrix}$$

$$X = a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v} \qquad Y = c\frac{\partial f}{\partial u} + d\frac{\partial f}{\partial v}$$