Exercise (PSet 3)

$$g = Jf^T Jf$$

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we often write $\quad ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad$ ds = arc length element

[equivalently, $\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$]

where $X = \dot\alpha(0) \qquad \alpha(t) = f(u(t), v(t))$

$$\langle X, X\rangle = \langle \dot\alpha(0), \dot\alpha(0)\rangle = \langle f_u\dot u + f_v\dot v, f_u\dot u + f_v\dot v\rangle$$

$$= E\,\dot u^2 + 2F\,\dot u\,\dot v + G\,\dot v^2$$

thus $\|X\| = \sqrt[2]{E\frac{du}{dt} + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}}$

so $s(t) = \int_a^b ds = \int_a^b \sqrt[2]{E\frac{du}{dt}^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}^2}\,dt = \int_a^b \|\dot\alpha(t)\|\,dt$

Upshot, g or I or $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

lets us measure the arc length of curves on S

also angle: the angle between two tangent vectors

X, Y is given by

$$\cos\theta = \frac{I(X,Y)}{\|X\|\,\|Y\|}$$

if $X = \frac{\partial f}{\partial u}$ $\quad Y = \frac{\partial f}{\partial v}$, then

$$\cos\varphi = \frac{F}{\sqrt{EG}} = 0 \iff F = 0$$

Examples:

~~[illegible]~~ 1) The sphere of radius 1 in spherical coordinates

$$f(\varphi, \theta) = (\cos\varphi \sin\theta,\ \sin\varphi \sin\theta,\ \cos\theta) \qquad \theta \in (0, \pi)$$

$$\varphi \in (0, 2\pi)$$

$$f_\varphi = (-\sin\varphi \sin\theta,\ \cos\varphi \sin\theta,\ 0)$$

$$f_\theta = (\cos\varphi \cos\theta,\ \sin\varphi \cos\theta,\ -\sin\theta)$$

$$\langle f_\varphi, f_\varphi \rangle = \sin^2\varphi \sin^2\theta + \cos^2\varphi \sin^2\theta = \sin^2\theta$$

$$\langle f_\varphi, f_\theta \rangle = 0 \qquad \langle f_\theta, f_\theta \rangle = \cos^2\varphi \cos^2\theta + \sin^2\varphi \cos^2\theta + \sin^2\theta = 1$$

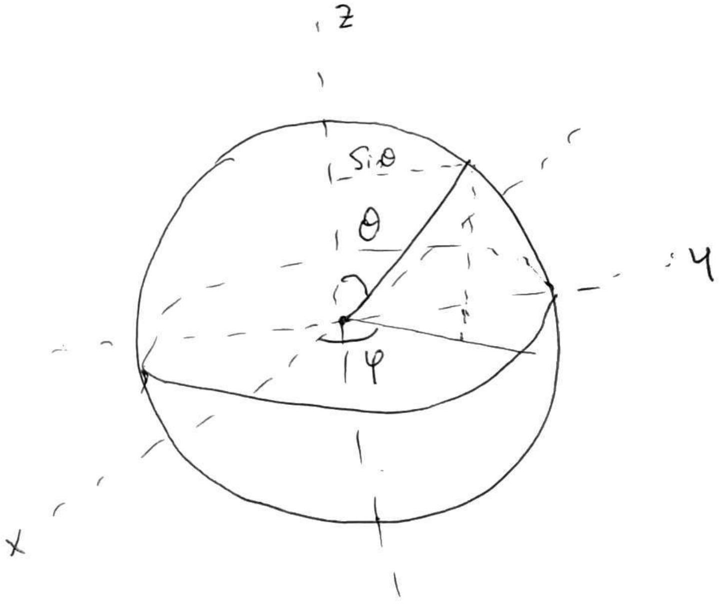
So

$$(g_{ij}) = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$ds^2 = \sin^2\theta \, d\varphi^2 + d\theta^2$$

$$\alpha(t) = f(\varphi(t), \theta(t))$$

$$s = \int \sqrt{\sin^2\theta \frac{d\varphi}{dt}^2 + \frac{d\theta}{dt}^2} \, dt$$



e.g. Consider a curve with constant $\theta = \theta_0$

& $\varphi(t) = t$ $\quad \alpha(t) = (\cos t \sin\theta_0, \sin t \sin\theta_0, \cos\theta_0)$

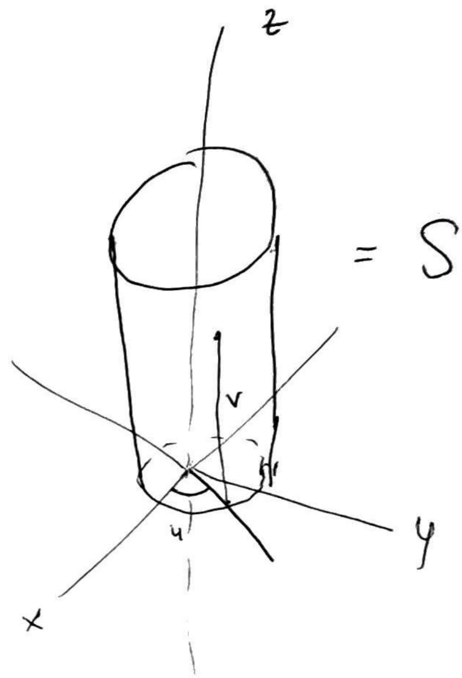
$t \in (0, l)$ $\quad 0 < 2\pi$

arc length $\quad ds^2 = \sin^2\theta_0 \, dt^2$

$$S(t) = \int \sqrt{\sin^2\theta_0} \, dt = (\sin\theta_0)\, t$$

note $\alpha(t)$ is exactly an arc of a lesser circle on S^2 of radius $\sin\theta_0$.

2) the right cylinder of radius 1



$f(u,v) = (\cos u, \sin u, v)$

$f_u = (-\sin u, \cos u, 0)$

$f_v = (0, 0, 1)$

$\langle f_u, f_u \rangle = \sin^2 u + \cos^2 u = 1$

$\langle f_u, f_v \rangle = 0 \quad \langle f_v, f_v \rangle = 1$

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

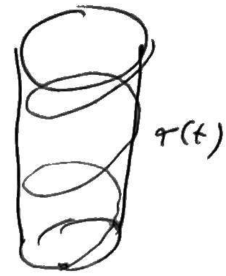
note this is the same inner product as $\mathbb{R}^2$!

This means all measurements of length & angle on the cylinder are "the same" as on the plane

"the cylinder is flat"

Consider e.g. the helix $\alpha(t) = (\cos t, \sin t, ht)$

$$\alpha : \mathbb{R} \to S$$



the arc length of α is given by

$$\int_a^b \sqrt{E \frac{du}{dt}^2 + 2F \frac{du}{dt}\frac{dv}{dt} + G \frac{du}{dt}^2}\, dt = \int_a^b \sqrt{1 + h^2}\, dt$$

$$= (b-a)\sqrt{h^2+1}.$$

<u>Rmk</u> In the previous two examples, $F = 0$

this means $\left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right\rangle = 0$ so the bases vectors

$\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$ of $T_p S$ are orthogonal. This isn't always the case

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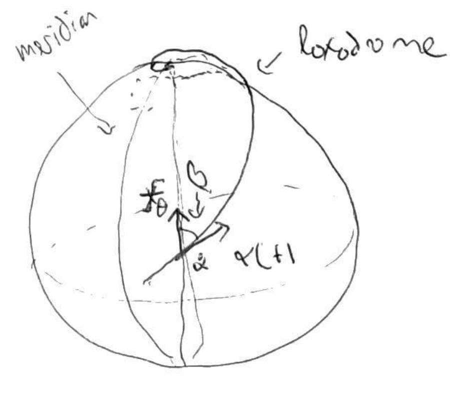
3) (loxodromes) lets consider the sphere again

with $(g_{ij}) = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix}$

loxodrome = curve w/ constant angle w/ meridian

"constant compass path = straight line on mercator projection"

meridian = curve w/ constant φ



meridians φ = constant have tangent

f_θ while $\dot{\alpha}(t) = \dot{\theta} f_\theta + \dot{\varphi} f_\varphi$