Using (g_{ij}), we compute (β = constant)

$$\cos(\beta) = \frac{\langle f_\theta, \dot\alpha \rangle}{\|f_\theta\| \, \|\dot\alpha\|} = \frac{\dot\theta}{1\sqrt{\sin^2\theta\,\dot\varphi^2 + \dot\theta^2}} = \text{Constant}$$

$$\iff \underbrace{A}_{\text{some constant}} \dot\theta^2 = \dot\varphi^2 \sin^2\theta \qquad \frac{\dot\theta}{\sin\theta} = \pm \frac{\dot\varphi}{A}$$

$$\log \tan\frac{\theta}{2} = \pm \frac{(\varphi + B)}{A}$$

A depends on β

B = constant of integration depends on the starting point.

Lemma Suppose $\tilde f = f \circ \varphi$ is another parametrization with metric $(\tilde g_{ij})$, then

$$(\tilde g_{ij}) = J\varphi^T (g_{ij}) J\varphi$$

PF clear from the exercise $(g_{ij}) = Jf^T Jf$ + chain rule

Surface integrals

(e.g. Suppose f is a chart for some surface U)

Def let $f : U \to \mathbb{R}^3$ be a surface element s.t. $f(U)$ is a regular surface & let α be a continuous function on $f(U)$, let $Q \subseteq U$ some closed + bounded (that is "compact") subset.

We define the surface integral

$$\iint_{f(Q)} \alpha \, dA := \iint_Q (\alpha \circ f)(u,v) \sqrt{\det(g_{ij})} \, du \, dv$$

when $\alpha \equiv 1$, $\displaystyle\iint_{f(Q)} dA =:$ Area of $f(Q)$.

<u>Claim</u> $\displaystyle\iint_{f(Q)} \alpha \, dA$ is independent of parametrization

<u>Pf</u> suppose $\tilde{f}(\tilde{Q}) = f(Q)$ for $\tilde{f}$ some other par

$\tilde{f} : \tilde{U} \to \mathbb{R}^3$

$\tilde{f} = f \circ \varphi$

$\varphi : \tilde{U} \xrightarrow{\sim} U$

$\varphi(\tilde{Q}) = Q$

$$\iint_{\tilde{f}(\tilde{Q})} \alpha \, dA = \iint_{\tilde{Q}} (\alpha \circ \tilde{f})(\tilde{u},\tilde{v}) \sqrt{\det(\tilde{g}_{ij})} \, d\tilde{u} \, d\tilde{v}$$

Lemma above

$$= \iint_{\tilde{Q}} (\alpha \circ \tilde{f}) \det J\varphi \sqrt{\det(g_{ij})} \, d\tilde{u} \, d\tilde{v}$$

$J\varphi \, d\tilde{u} \, d\tilde{v}$

$du \, dv$

$$= \iint_{\tilde{Q}} (\alpha \circ f \circ \varphi) \det J\varphi \sqrt{\det(g_{ij})} \, d\tilde{u} \, d\tilde{v}$$

Change of variables formula

$$= \iint_Q (\alpha \circ f) \sqrt{\det(g_{ij})} \, du \, dv$$

$$= \int_{f(Q)} \alpha \, dA$$

$\blacksquare$

note

$$\det(g_{ij}) = \left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\|^2 = EG - F^2$$

So the area form $dA = \sqrt{EG - F^2}\, du\, dv$

and so area of a region $R \subseteq S$ contained in a chart $f : U \longrightarrow S$ is

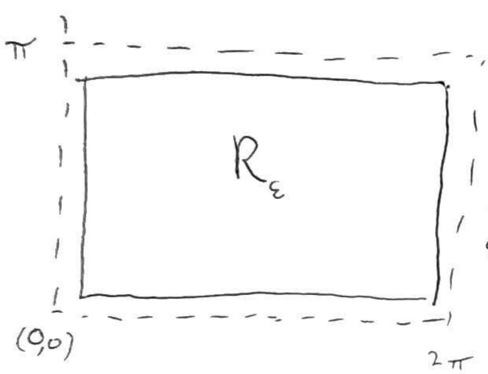
$$\iint_{f^{-1}(R)} \sqrt{EG - F^2}\, du\, dv$$

<u>Ex</u> 1) Area of a sphere of radius r

$f(u,v) = (r\cos\varphi \sin\theta, r\sin\varphi \sin\theta, r\cos\theta)$ $\qquad g_{ij} = \begin{pmatrix} r^2\sin^2\theta & 0 \\ 0 & r^2 \end{pmatrix}$

$U = (0, 2\pi) \times (0, \pi)$ $\qquad \deg(g_{ij}) = r^4 \sin^2\theta \qquad dA = r^2 \sin\theta\, d\theta\, d\varphi$

$R_\varepsilon = [\varepsilon, 2\pi - \varepsilon] \times [\varepsilon, \pi - \varepsilon]$



$\mathrm{Area}(S^2) =$

$$\lim_{\varepsilon \to 0} \mathrm{Area}(f(R_\varepsilon)) = \iint_{R_\varepsilon} r^2 \sin\theta\, d\theta\, d\varphi =$$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^{2\pi - \varepsilon} r^2 \left(\int_\varepsilon^{\pi - \varepsilon} \sin d\theta \right) d\varphi = 4\pi r^2$$

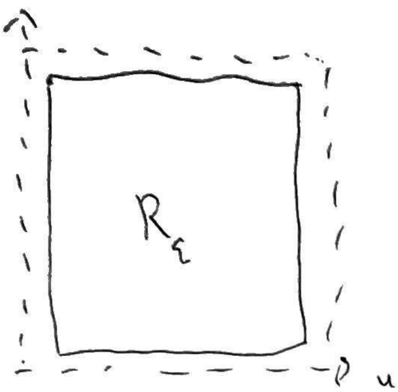
note that the complement of $f(U)$ in S^2 doesn't contribute to the area so we can ignore it.

Ex 2

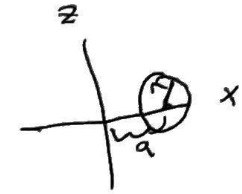
The torus

$$f(u,v) = \big((a + r\cos u)\cos v,\ (a + r\cos(u))\sin v,\ r\sin u\big)$$

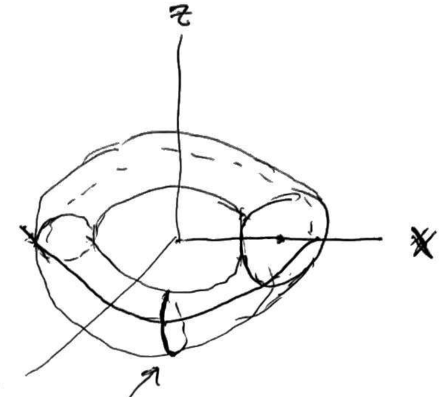
$u \in (0, 2\pi)$ $\quad v \in (0, 2\pi)$ $\qquad R_\varepsilon = [\varepsilon, 2\pi - \varepsilon] \times [\varepsilon, 2\pi - \varepsilon]$



Surface of revalution of the circle $(a + r\cos u, r\sin u)$ in the (x,z) plane



$T =$



$f(U) = T \setminus$ this curve

$$f_u = (-r\sin u \cos v,\ -r\sin u \sin v,\ r\cos u)$$

$$f_v = \big(-(a + r\cos u)\sin v,\ (a + r\cos u)\cos v,\ 0\big)$$

$E = \langle f_u, f_u \rangle = r^2 \qquad F = \langle f_u, f_v \rangle = 0 \qquad G = \langle f_v, f_v \rangle = (a + r\cos u)^2$

$$dA = \sqrt{EG - F^2}\, du\, dv = (r^2\cos u + ra)\, du\, dv$$

$$\text{Area}(T) = \lim_{\varepsilon \to 0} A(R_\varepsilon) = \lim_{\varepsilon \to 0} \iint_{f(R_\varepsilon)} dA = \lim_{\varepsilon \to 0} \iint_{R_\varepsilon} (r^2\cos u + ra)\, du\, dv$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^{2\pi - \varepsilon} \left[\int_\varepsilon^{2\pi - \varepsilon} (r^2\cos u + ra)\, du\right] dv = 2\pi\Big[r^2\sin u + rau\Big]_{u=0}^{2\pi} = 4\pi^2 ra$$

Def let S be a regular surface, a vector field
$\underset{\mathbb{R}^3}{\cap}$
is a continuously diff map
(or C^n or C^∞)

$$X : S \longrightarrow \mathbb{R}^3$$

Here we identify $\mathbb{R}^3$ with $T_p\mathbb{R}^3$ for ~~each~~ $p \in S$ &
view $X(p) \in T_p\mathbb{R}^3$

A tangent vector field is a vector field X s.t.

$$X(p) \in T_pS \subseteq T_p\mathbb{R}^3 \quad \text{for each } p \in S$$

A normal vector field is a vector field s.t.

$$X(p) \in T_pS^\perp \subseteq T_p\mathbb{R}^3 \quad \text{for all } p \in S$$

If $f\ U \longrightarrow S \subseteq \mathbb{R}^3$ is a chart,
we can write a tangent vector field as

$$X(u,v) = \alpha(u,v)\frac{\partial f}{\partial u} + \beta(u)\frac{\partial F}{\partial v}$$

Similiarly, a normal vector field is written

$$X(u,v) = \gamma(u,v)\ \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}$$

Where α, β, γ are continuously differentiable
(or C^n or C^∞)

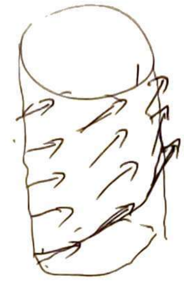
Ex 1) $f(\varphi, x) = (\cos\varphi, \sin\varphi, x)$ is a parametrized cylinder

the vector field

$$X(\varphi, x) = (-\sin\varphi, \cos\varphi, a) \quad a \text{ constant}$$

is a tangent vector field. In fact its the vector field of tangents to the helices

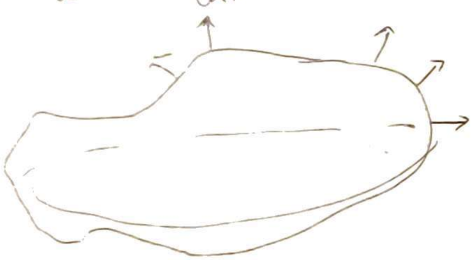
$$t \overset{\alpha_c}{\longmapsto} (\cos(t), \sin(t), at + c)$$



2) consider $f : U \to S \subseteq \mathbb{R}^3$

$$\nu = \pm \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\|}$$

is a unit normal vector field



Can view ν as a map ~~[illegible]~~ $\nu : U \to S^2 \subseteq \mathbb{R}^3$ the unit sphere

So called Gauss Map

Question Does the Gauss map extend to all of S?

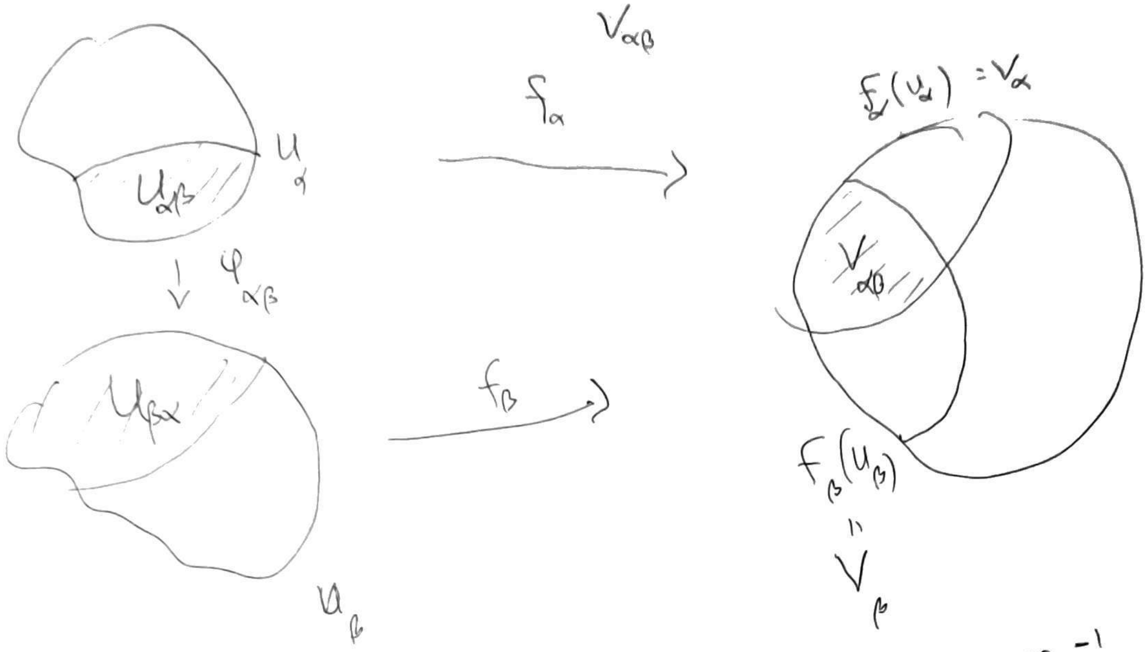
Ans Not always!

Def A surface $S \subseteq \mathbb{R}^3$ (Resp hypersurface $\subseteq \mathbb{R}^n$) is called orientable if it can be covered by charts

$(f_\alpha : U_\alpha \to S)_{\alpha \in I}$ s.t. $\varphi_{\alpha\beta} := f_\beta^{-1} \circ f_\alpha : U_{\alpha\beta} \to U$

has positive determinant. Here $(f_\alpha : U_\alpha \to V_\alpha \subseteq S)$

$$U_{\alpha\beta} = f_\alpha^{-1}\big(\underbrace{\overset{V_\alpha}{\overset{\shortparallel}{f_\alpha(U_\alpha)}} \cap \overset{V_\beta}{\overset{\shortparallel}{f_\beta(U_\beta)}}}_{\overset{\shortparallel}{V_{\alpha\beta}}}\big)$$



$\varphi_{\alpha\beta}$ are called <u>transition maps</u>

Note $\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha}$

& $\varphi_{\alpha\beta}$ is a diffeomorphism

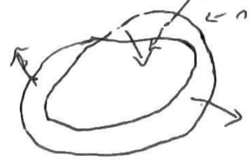
<u>Rmk</u> If S is orientable, the surface element $dA = \sqrt{g}\, du\, dv$ is well defined globally on S

<u>Lemma</u> $S \subseteq \mathbb{R}^3$ is orientable $\Leftrightarrow$ $\exists$ a continuous unit normal vector field

$$\nu : S \to \coprod_p T_p S^\perp \quad \left(= \text{"}NS\text{" the normal bundle of } S\right)$$

locally, $\nu = \pm \left(\dfrac{f_u \times f_v}{\|f_u \times f_v\|}\right)$ $p \in S$

<u>Ex</u> Mobius strip is not orientable



← not continuous!

Will compute this on HW 4