Math 136 — Differential geometry **Due**
Problem Set 1 **Friday, September 17**

Problem 1. Let $c(t)$ be an arbitrary parametrization of a Frenet curve in $\mathbb{R}^3$. Show that the curvature and torsion are computed by the following formulas.

$$\kappa(t) = \frac{\|\dot{c}\times\ddot{c}\|}{\|\dot{c}\|^3} \quad \tau(t) = \frac{\det(\dot{c},\ddot{c},\dddot{c})}{\|\dot{c}\times\ddot{c}\|^2}$$

You may need to use the chain rule to relate the $\dot{c}$ to the derivative with respect to arc length.

Problem 2. Show that for any two points $p, q \in \mathbb{R}^n$, a regular curve with minimal distance from p to q is necessarily the straight line segment from p to q. You may need to use the Cauchy-Schwarz inequality

$$\langle v, w\rangle \leq \|v\| \cdot \|w\| \text{ with equality if and only if } v \text{ and } w \text{ are parallel}$$

applied to the tangent vector and the difference $p - q$.

Problem 3. Let c be a regular parametrized plane curve. The *evolute* of c is the curve traced out by the centers of the osculating circles of c. Explicitly, we can write the evolute as a parametrized curve

$$z = c + \frac{1}{\kappa}e_2.$$

(1) Show that the evolute is a regular curve if and only if $\kappa' \neq 0$.
(2) Show that for each t, the tangent vector to the evolute is perpendicular to the curve c.

Problem 4. The *catenary* is the plane curve which traces out the shape of a horizontal rope hanging under its own weight. It is given by the equation

$$c(t) = (t, \cosh(t)).$$

(1) Show that the curvature of c is given by
$$\kappa(t) = \frac{1}{\cosh^2(t)}.$$
(2) Show that the evolute of c is given by
$$z(t) = (t - \sinh(t)\cosh(t), 2\cosh(t)).$$
Is the evolute a regular curve?

Recall that the functions $\cosh(t)$ and $\sinh(t)$ are defined as

$$\cosh(t) = \frac{e^t+e^{-t}}{2} \quad \sinh(t) = \frac{e^t-e^{-t}}{2}$$

Problem 5. Suppose all the normal lines to a space curve c pass through a fixed point. Show that the curve is contained in a circle.