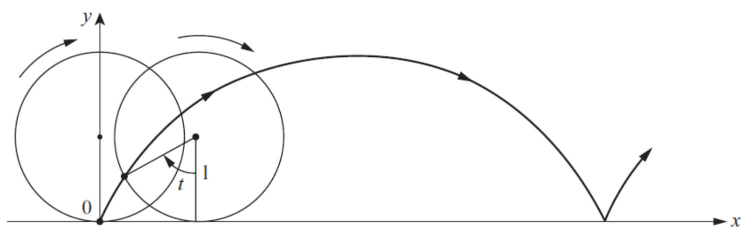
Math 136 — Differential geometry **Due**
Problem Set 2 **September 24**

Problem 1. Consider the ellipse parametrized by $c(t) = (a\cos t, b\sin t)$ with $a \neq b$. Find the *vertices* of the ellipse, i.e. the local extrema of the function $\kappa(t)$.



Problem 2. The *cycloid* is the curve traced out by a fixed point on a circle as the circle rolls along a straight line (figure above).

(1) Find a parametrization of the cycloid.
(2) Compute the curvature of the cycloid.

Problem 3. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve parametrized by arc length. Show that for any point $s_0 \in s$, the curvature of γ at s_0 is equal to the curvature of the projection of $\pi \circ \gamma : I \to \mathbb{R}^2$ at s_0 where π is the projection onto the osculating plane.

Problem 4. Consider a plane curve give in polar coordinates (r, θ) by the equation $r = r(\theta)$ and denote by $r' = \frac{dr}{d\theta}$.

(1) Show that the arc length from θ_1 to θ_2 can be calculated as
$$\int_{\theta_1}^{\theta_2} \sqrt{r'^2 + r^2}\, d\theta.$$
(2) Show that the curvature is given by
$$\kappa(\theta) = \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}}.$$
(3) Calculate the curvature for the Archimedean spiral given by $r(\theta) = a\theta$.

Problem 5. Let $c : I \to \mathbb{R}^3$ be a Frenet curve with nonzero torsion τ and consider the unit normal vector $e_2(s)$, called the *principal normal vector*. We say that c is a *Bertrand curve* if there exists a scalar function r such that the curve
$$\bar{c}(s) := c(s) + r(s)e_2(s)$$
has the same principal normal vector as $c(s)$, namely $e_2(s)$. In this case, we say c and $\bar{c}$ are a Bertrand pair. Suppose c and $\bar{c}$ are a Bertrand pair.

(1) Show that $r(s)$ is constant. Conclude that the distance between c and $\bar{c}$ is also constant.
(2) Show that the angle between the tangent vectors of c and $\bar{c}$ is constant.
(3) Show that there exist constants a and b such that $a\kappa + b\tau \equiv 1$ where κ and τ are the curvature and torsion of c.
(4) Give an example of a Bertrand pair.