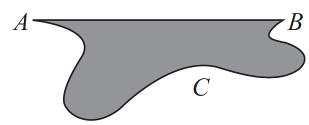
Math 136 — Differential geometry | **Due**
Problem Set 3 | **October 1**

Problem 1. Let $c : I \to \mathbb{R}^n$ be an arc-length parametrized Frenet curve. Show that

$$\det(c', c'', \ldots, c^{(n)}) = \prod_{i=1}^{n-1} (\kappa_i)^{n-i}.$$



Problem 2. Consider a line segment L between points A and B and let $l > \text{length}(L)$. Let C be a curve of length l passing through A and B such that $L \cup C$ is a simple closed curve (see the picture above). Show that the curve C such that $L \cup C$ bounds the largest possible area is an arc of a circle.

Problem 3. Recall Green's theorem: let C be a positively oriented simple closed curve bounding a region D in the plane and let L, M be two C^1 functions, then

$$\oint_C L dx + M dy = \iint_D (M_x - L_y) dx dy.$$

Use this to show that

$$A = \frac{1}{2} \int_a^b (xy' - yx') dt$$

where $\alpha : [a, b] \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$ is a parametrization of C and A is the area of D.

Problem 4. Let $f : U \to \mathbb{R}^n$ be a regular parametrized hypersurface. Show that the matrix (g_{ij}) of the first fundamental form can be written as

$$(g_{ij}) = (Jf)^T Jf$$

where Jf is the Jacobian matrix of f.

Problem 5. A surface of revolution is obtained by rotating a regular curve in the $x - z$ plane around the z axis; it can be parametrized by

$$f(t, \theta) = (r(t) \cos\theta, r(t) \sin\theta, h(t))$$

where the curve in the $x - z$ plane is given by $(r(t), h(t))$. Compute the first fundamental form of f.