Math 136 — Differential geometry **Due**
Problem Set 4 **October 8**

Problem 1. Consider the parametrized surface element $f(u, v) = (au \cos v, bu \sin v, u^2)$. Draw a picture of the image of f and compute its first fundamental form.

Problem 2. Consider the parametrized surface element

$$f(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

(1) Let S^2 be the sphere $x^2 + y^2 + z^2 = 1$. Show that $f(u, v)$ is a regular parametrization of $S^2 \setminus (0, 0, 1)$ which is bijective (and thus gives a chart exhibiting $S^2 \setminus (0, 0, 1)$ as a regular surface).
(2) Compute the first fundamental form (g_{ij}) of f.
(3) Use this to compute the surface area

$$\iint_{S^2} dA.$$

Problem 3. Consider the Möbius strip

$$f(u,v) = \left[\sin(u)\left(1 + v\sin(\frac{u}{2})\right), \cos(u)\left(1 + v\sin(\frac{u}{2})\right), v\cos(\frac{u}{2})\right]$$

where $(u, v) \in [0, 2\pi] \times (-1, 1)$.

(1) Show that $f(0, v) = f(2\pi, -v)$ and conclude that $f|_{\mathbb{R}\times\{0\}}$ is a simple closed curve.
(2) Compute the normal vector field

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}.$$

(3) Conclude that the Möbius strip is not orientable.

Problem 4. Show that the area of a compact region Q of graph $z = f(x, y)$ is given by

$$\iint_{\pi(Q)} \sqrt{1 + f_x^2 + f_y^2} dxdy$$

where π is the projection onto the (x, y) plane.

Problem 5. Let $g : S \to \mathbb{R}$ be a differential function. The gradiant ∇g is defined as the unique vector field satisfying

$$\langle \nabla g(p), v \rangle = dg_p(v)$$

for any p and $v \in T_pS$. Let $f : U \to S$ be a chart of $p \in S$.

(1) Show that

$$\nabla g = \frac{g_u G - g_v F}{EG - F^2} f_u + \frac{g_v E - g_u F}{EG - F^2} f_v.$$

(2) Let $C = \{g(q) = \text{constant}\}$ be a level curve. Suppose $\nabla g \neq 0$ for all $p \in C$. Show that C is a regular curve and that ∇g is normal to C.