Math 136 — Differential geometry **Due**
Problem Set 5 **Oct 15**

Problem 1. Let $c : I \to S \subset \mathbb{R}^3$ be a curve contained in a regular surface S.

(1) Show that the curvature κ of c satisfies
$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$
where κ_g is the geodesic curvature and κ_n is the normal curvature.

(2) Suppose that S has positive Gaussian curvature, $K > 0$. Show that κ satisfies
$$\kappa \geq \min\{|\kappa_1|, |\kappa_2|\}$$
where κ_i are the principal curvatures of S.

Problem 2. Fix a point $p \in S$ a regular surface. Let θ be a parametrization of the unit tangent directions at p, (e.g. if e_1, e_2 is an orthonormal basis of T_pS and we parametrize the unit tangents as $e_1 \cos\theta + e_2 \sin\theta$ for $\theta \in [0, 2\pi]$). Show that the mean curvature satisfies
$$H(p) = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta) d\theta$$
where $\kappa_n(\theta)$ is the unit normal curvature in the direction of θ.

Problem 3. Consider the *catenoid* parametrized by
$$f : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3 \qquad f(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$
Compute the following:

(1) the first fundamental form (g_{ij}),
(2) the second fundamental form (h_{ij}),
(3) the principal curvatures κ_i,
(4) the Gaussian curvature K and mean curvature H.

Problem 4. Suppose that a regular surface $S \subset \mathbb{R}^3$ is tangent to a plane $H \subset \mathbb{R}^3$ along a regular curve C. That is, $H \cap S = C$ and $H = T_pS$ for all $p \in C$. Show that every point of S along C is parabolic or a level point.

Problem 5. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $S = \{z = g(x, y)\} \subset \mathbb{R}^3$ be the graph of g.

(1) Compute the first and second fundamental forms g_{ij} and h_{ij}.

(2) Let D^2g denote the Hessian matrix of g, that is, the matrix of second partial derivatives of g, and let Dg denote the Jacobian matrix of g. Show that the Gaussian curvature of S is given by
$$K = \frac{\det D^2 g}{(1 + \|Dg\|^2)^2}.$$

(3) Compute the mean curvature H of S.

(4) Recall that for every regular surface S and $p \in S$, there exist coordinates such that S is the graph of a function g locally around p. Argue that these coordinates can be chosen so that $g(0, 0) = 0$ and that $Dg(0, 0) = (0, 0)$. Conclude that in this case that at the point p, $(h_{ij}) = D^2g$, $K = \det D^2 g$ and $H = \frac{1}{2}\mathrm{Tr} D^2 g$.