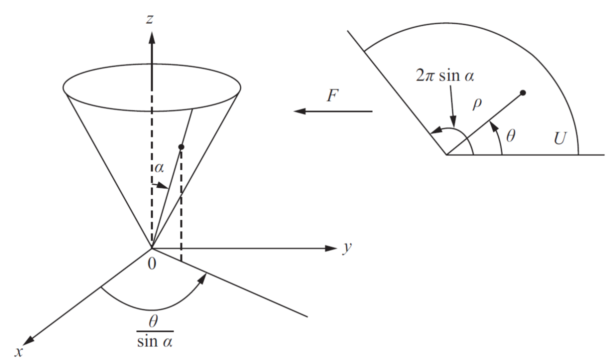
Math 136 — Differential geometry **Due**
Problem Set 7 **November 5**

Problem 1. Let $S \subset \mathbb{R}^{n+1}$ be a regular hypersurface and let ∇ denote the covariant derivative. Suppose X, X_1, X_2, Y, Y_1 and Y_2 are smooth tangent vector fields to S and f, f_1 and f_2 smooth (scalar) functions. Show the following properties for the covariant derivative:

(1) (linearity) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$;
(2) (additivity) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;
(3) (product rule) $\nabla_X (fY) = (\nabla_X f) Y + f \nabla_X Y$
(4) (compatibility with the metric) $\nabla_X \langle Y_1, Y_2 \rangle = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle$



Problem 2. Consider the cone C and circular wedge S (pictured above on the left and right respectively). Here α is the constant cone angle which corresponds to the wedge having angle $2\pi \sin \alpha$. Consider S as a surface in $\mathbb{R}^3$ lying in the (x, y)-plane.

(1) Write down a parametrization for S in terms of ρ and θ.
(2) Write down a parametrization of an open subset of C in terms of ρ and θ.
(3) Compute the first fundamental form of S and C in these coordinates.
(4) Conclude that S and C are locally isometric away from the cone point.

Problem 3. Suppose $f : U \to S \subset \mathbb{R}^3$ is a chart such that the metric takes the form

$$(g_{ij}) = \lambda(u, v) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for a smooth function λ. Such a chart is called a *conformal parametrization* or *isothermal coordinates*. Show that in such coordinates, the Gauss curvature can be written as

$$K = -\frac{1}{2\lambda} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \lambda$$

Problem 4. Show that no point of the sphere has a neighborhood isometric to a neighborhood of the plane.

Problem 5. Show that there does not exist a regular surface in $\mathbb{R}^3$ with the following metric and second fundamental forms.

$$g_{ij}(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad h_{ij}(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$