Math 136 — Differential geometry **Due**
Problem Set 8 **Nov 12**

Problem 1. Fix a point $p \in S$ and let C be a curve through p. Show that the geodesic curvature $\kappa_g(p)$ of C at p is equal to the curvature $\kappa(p)$ at p of the normal projection of C onto the tangent plane T_pS.

Problem 2. Suppose S is a surface with metric g and suppose that we have isotherman coordinates (u^1, u^2), where the metric takes the form

$$(g_{ij}) = \lambda(u^1, u^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\lambda > 0$ is a non-negative smooth function.

(1) Write down the geodesic equations in this chart.
(2) Consider the special case $S = \{u^2 > 0\} \subset \mathbb{R}^2$ and $\lambda = u^2$. Note here this metric may not necessarily arise from a parametrization of a surface in $\mathbb{R}^3$ but we can still compute with it as an abstract metric. Compute the geodesic $\gamma(t) = (u^1(t), u^2(t))$ with initial condition $\gamma(0) = (a, b)$ and $\dot{\gamma}(0) = (0, \frac{1}{\sqrt{b}})$, $b > 0$.

The *Poincaré upper half plane* is $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the abstract metric

$$g = \frac{1}{y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 3. Show that the geodesic equations for the upper half plane can be written as

$$\frac{d}{ds}\frac{x'}{y^2} = 0 \qquad \frac{d^2y}{ds^2} = \frac{1}{y}((x')^2 - (y')^2)$$

where $\gamma(s) = (x(s), y(s))$ is an arc-length parametrized curve.

Problem 4. Show that the geodesics on the upper half plane are vertical lines as well as semi-circles centered at a point on the x-axis. Hint: what is the equation of a circle with center $(b, 0)$ and radius r?

Problem 5. Calculate the Gaussian curvature K of the upper half plane.