Math 136 — Differential geometry **Due**
Problem Set 9 **Nov 22**

Problem 1. Consider the upper half plane $\mathbb{H} = \{(x, y) \mid y > 0\}$ with metric

$$g = \frac{1}{y^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as in problem Set 8. Recall the Riemannian curvature tensor R is defined by the formula

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for tangent vector fields X, Y and Z.

(1) Compute the coefficients ${R^s}_{jkl}$ of the curvature tensor defined by the basis expansion

$$R\left(\frac{\partial f}{\partial u^j}, \frac{\partial f}{\partial u^k}\right)\frac{\partial f}{\partial u^l} = {R^s}_{jkl}\frac{\partial f}{\partial u^s}.$$

(2) Compute the coefficients of curvature with lowered indices: $R_{ijkl} := g_{is}{R^s}_{jkl}$.
(3) Compute the *Ricci curvature*, defined as the tensor with coefficients $R_{jl} = g^{ik}R_{ijkl}$.
(4) Compute the *scalar curvature*, defined as the scalar function $S = g^{jl}R_{jl}$ where R_{jl} is the Ricci curvature.

Problem 2. Let $f : U \to S \subset \mathbb{R}^{n+1}$ be a chart for a regular surface. Show that the Christoffel symbol of the second kind can be expressed as

$$\Gamma^k_{ij} = g^{kl}\left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l}\right\rangle.$$

Recall that we computed the following formula for Γ^k_{ij} in class:

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(-g_{ij,l} + g_{jl,i} + g_{li,j}).$$

Problem 3. Show that the Gauss equation can be written as

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$$

where R_{ijkl} is the lower index Riemannian curvature tensor. Conclude that the Theorem Egregium in dimension 2 is equivalent to the statement

$$K = \frac{R_{1212}}{\det g}$$

Problem 4. We will define the covariant derivative of the shape operator, $\nabla_X L$, via the formula

$$(\nabla_X L)Y = \nabla_X(LY) - L(\nabla_X Y).$$

Note this is designed so that there is a "product rule" $\nabla_X(LY) = (\nabla_X L)Y + L(\nabla_X Y)$. Define the coefficients $\nabla_i h^j_k$ of ∇L via the formula

$$\left(\nabla_{\frac{\partial f}{\partial u^i}} L\right)\frac{\partial f}{\partial u^k} = \nabla_i h^j_k \frac{\partial f}{\partial u^j}.$$

Show that the Codazzi-Mainardi equation is equivalent to the equation

$$\nabla_i h^j_k = \nabla_k h^j_i.$$