Lecture 1: Overview

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1 Introduction

One of the characterizing features of algebraic geometry is that the set of all geometric objects of a fixed type (e.g. smooth projective curves, subspaces of a fixed vector space, or coherent sheaves on a fixed variety) often itself has the structure of an algebraic variety (or more general notion of algebro-geometric space). Such a space $\mathcal{M}$ is the *moduli space* classifying objects of the given type and in some sense the study of all objects of the given type is reduced to the studying the geometry of the space $\mathcal{M}$. This self-referential nature of algebraic geometry is a crucial aspect of the field.

More precisely, suppose we are interested in studying some class of geometric objects $\mathcal{C}$ with a suitable notion of a family of objects of $\mathcal{C}$

$$\pi : \mathscr{X} := \bigsqcup \{X_b \in \mathcal{C} \ : \ b \in B\} \to B$$

parametrized by some base scheme B. To a first approximation, we may attempt to construct a moduli space for the class $\mathcal{C}$ in two steps. First we find a family $\pi : \mathscr{X} \to B$ such that for each object $X \in \mathcal{C}$, there exists a $b \in B$ with $X_b \cong X$. Next we look for an equivalence relation on B such that $b \sim b'$ if and only if $X_b \cong X_{b'}$ and such that the quotient of B by this equivalence relation inherits the structure of an algebraic variety. If this happens, we may call $\mathcal{M} := B/\sim$ a moduli space and the family of objects $\mathcal{M}$ inherits from π the *universal family* (we will discuss this more carefully soon). In particular, the points of $\mathcal{M}$ are in bijection with isomorphism classes of objects in $\mathcal{C}$.

Moduli spaces give a good answer to the question of classifying algebraic varieties, or more generally objects of some class $\mathcal{C}$. In the best case scenario, we may have that

$$\mathcal{M} = \bigsqcup_{\Gamma} \mathcal{M}_\Gamma$$

where d is some discrete invariant (not necessarily an integer) and each component $\mathcal{M}_\Gamma$ is of finite type. Then classifying the objects of $\mathcal{C}$ reduces to (1) classifying the discrete invariants Γ, and (2) computing the finite type spaces $\mathcal{M}_\Gamma$.

Example 1. *The prototypical example which we will discuss at length later in the class is that of smooth projective curves.*[1] *Here there is one discrete invariant, the genus g, and the moduli space is a union*

$$\mathcal{M} = \bigsqcup_{g \in \mathbb{Z}_{\geq 0}} \mathcal{M}_g$$

[1] For us a curve is a finite type k-scheme with pure dimension 1 for k a field.

of smooth $3g-3$-dimensional components. This example was originally studied by Riemann in his 1857 *paper* Theorie der Abel'schen Functionen *where he introduced the word* moduli *to refer to the $3g-3$ parameters that (locally) describe the space $\mathcal{M}_g$.*

1.1 Facets of moduli theory

This class will focus on the following three facets of moduli theory.

Existence and construction

Hilbert schemes, algebraic stacks, GIT, Artin algebraization, coarse and good moduli spaces

Compactifications

Semi-stable reduction, Deligne-Mumford-Knudsen-Hassett compactifications, wall-crossing, KSBA stable pairs

Applications

Enumerative geometry and curve counting, computing invariants, constructing representations, combinatorics, arithmetic statistics

Of course we won't have time to cover everything written above (and there are countless more topics that fit under each heading) but I hope to give a feeling of the techniques and tools employed in moduli theory as well as the far reaching applications.

1.2 A note on conventions

For most of the class we will be working with finite type (or essentially of finite type) schemes over a field. I will make an effort to make clear when results require assumptions on the field (algebraically closed, characteristic zero) or when we work over a more general base. Not much will be lost if the reader wishes to assume everything is over the complex numbers throughout.

2 Motivating examples

Before diving in, I want to give some motivating examples of works that crucially relied on the tools and techniques of moduli theory. Many of the moduli theoretic ideas that come up in these examples will be discussed through the course.

2.1 Counting rational curves on K3 surfaces

Recall that a *K3 surface* is a smooth projective surface X with trivial canonical sheaf

$$\omega_X := \Lambda^2 \Omega_X \cong \mathcal{O}_X$$

and $H^1(X, \mathcal{O}_X) = 0$. A *polarized* K3 surface is a pair (X, H) where X is a K3 surface and H is an ample line bundle.

It turns out that for each g, there is a moduli space

$$\mathcal{M}_{2g-2}$$

parametrizing polarized K3 surfaces with $c_1(H)^2 = 2g - 2$.[2] The linear series $|H|$[3] is g-dimensional and the curves in $|H|$ have genus g. In particular one expects finitely many rational curves in $|H|$[4].

Let $n(g)$ denote the number of rational curves in $|H|$ for a generic polarized complex K3 surface $(X, H) \in \mathcal{M}_{2g-2}$. Note that the existence of a moduli space $\mathcal{M}_{2g-2}$ allows us to define generic as "corresponding to a point that lies in some Zariski open and dense subset of $\mathcal{M}_{2g-2}$." Then we have the following formula, conjectured by Yau and Zaslow, and proved by Beauville.

Theorem 1 (Beauville-Yau-Zaslow)**.**

$$1 + \sum_{g \geq 1} n(g) q^g = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}}$$

In particular, the numbers $n(g)$ are constant for general (X, H).

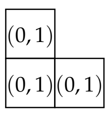
The proof here uses, among other things, a careful study of the compactified Jacobians of the (necessarily singular!) rational curves in $|H|$ and Hilbert schemes of points on X, two topics we will visit later in the class.

2.2 The $n!$-conjecture

A partition of n, denoted $\lambda \vdash n$, is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ with

$$\sum \lambda_i = n.$$

We can represent λ by a *Young diagram* of left aligned rows of boxes where the i^{th} row has λ_i boxes. Each box inherits a coordinate $(a, b) \in \mathbb{N}^2$ recording its position. In particular, the diagram has n boxes. For example, the partition $2 + 1 = 3$ corresponds to the following diagram.



We can list the n boxes $(a_1, b_1), \ldots, (a_n, b_n)$ of the diagram and then consider the matrix

$$\begin{bmatrix} x_1^{a_1} y_1^{b_1} & x_2^{a_1} y_2^{b_1} & \ldots & x_n^{a_1} y_n^{b_1} \\ \vdots & & \ddots & \vdots \\ x_1^{a_n} y_1^{b_n} & x_2^{a_n} y_2^{b_n} & \ldots & x_n^{a_n} y_n^{a_n} \end{bmatrix}$$

where the x_i and y_i are $2n$ indeterminates. Finally, let Δ_λ be the determinant of the above matrix. Note that Δ_λ is a homogeneous polynomial in both the x_i variables and the y_i variables. Furthermore, S_n acts by permuting the x_i and the y_i, and under this action, S_n acts on Δ_λ by the sign representation. In particular, Δ_λ is well defined up to a sign.

[2]Recall that $c_1(H)^2$ may be defined as the degree of $\mathcal{O}_X(C)|_C$ where C is the vanishing of a section of H.

[3]Recall the linear series of H is the space of divisors linearly equivalent to H, or equivalently, the projectivization $\mathbb{P}(H^0(X, H))$.

[4]Recall C is a rational curve if its normalization has genus 0.

Finally consider the vector space

$$D_\lambda := k[\partial_x, \partial_y]\Delta_\lambda$$

spanned by all partial derivatives of Δ_λ. This space carries a natural action of S_n. The $n!$ *conjecture*, proposed by Haiman and Garsia and later proved by Haiman, states the following.

Theorem 2 (Haiman). *D_λ as an S_n representation is isomorphic to the regular representation. In particular,* $\dim_k D_\lambda = n!$.

In our example partition above, we have the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

with determinant

$$\Delta_\lambda = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1.$$

The partial derivatives of Δ_λ are itself, constants, and the following linear forms.

$$\begin{aligned} x_3 - x_2, &\quad y_2 - y_3 \\ x_1 - x_3, &\quad y_3 - y_1 \\ x_2 - x_1, &\quad y_1 - y_2 \end{aligned}$$

The two columns above each span a copy of the standard two dimensional representation, Δ_λ spans the sign, and the constants span the trivial representation.

The theorem is proved by a careful study of the geometry of the Hilbert scheme of points on $\mathbb{A}^2$ which we will study in depth later in the class. In fact, if we denote by H_n the Hilbert scheme of n points in $\mathbb{A}^2$, then the $n!$ Theorem is equivalent to a particular moduli space $X_n \to H_n$ lying over H_n being Gorenstein![5]

Remark 1. *The motivation for the $n!$ conjecture came from symmetric function theory, and in particular, Macdonald positivity which is a corollary. In fact Macdonald positivity also has an interesting interpretation in terms of Hilbert schemes of points, and more precisely, the McKay correspondence for S_n acting on $\mathbb{A}^{2n}$. We will revisit this later.*

2.3 Alterations

Let X be a reduced locally Noetherian scheme over a field k. In many arguments it is useful to be able to replace X with a regular scheme that is "very close" to X. More precisely, a *resolution of singularities* of X is a morphism $f : X' \to X$ such that

- X' is regular,
- f is proper [6], and
- f is birational[7].

[5]This X_n is called the isospectral Hilbert scheme.
[6]This rules out the trivial operation of taking X' to be the regular locus of X
[7]This is one possible meaning of X' being very close to X.

Hironaka famously showed that when k has characteristic 0, resolutions of singularities always exist. The characteristic p case remains open. However, we have a positive result if we weaken the notion of being "very close" to X.

We say that $f : X' \to X$ is an *alteration* if it is proper, surjective, and generically finite. Then de Jong proved the following theorem which serves as a suitable replacement of Hironaka's theorem for many applications.

Theorem 3 (de Jong)**.** *Let X be a variety over a field k. Then there exists an alteration $f : X' \to X$ with X' regular.*

The proof of de Jong's theorem crucially uses the existence and properness of the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ by pointed stable curves. The following diagram gives a very basic sketch of the ideas involved.

$$
\begin{array}{ccccccc}
X''' & \xrightarrow[(5)]{} & X'' & \longrightarrow & X' & \longrightarrow & X \\
 & & \downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (0)} \\
 & & Y'' & \xrightarrow[(3)]{} & Y' & \xrightarrow[(1)]{} & Y
\end{array}
$$

Here, after possibly replacing X by some blowup, we find a projection (0) of relative dimension 1 with regular generic fiber. Then after taking an alteration (1) of Y, we can construct an alteration $X' \to X$ so that (2) has as fibers curves with at worst nodal singularities. Producing this map (2) with such properties is precisely where the existence of the Deligne-Mumford compactification is used! Then by induction on dimension, we have an alteration (3) with Y'' regular. Now the pullback (4) is a morphism with at worst nodal fibers over a regular base. In this situation X'' has nice singularities that can be explicitly resolved by (5) to obtain a regular X''' with the composition $X''' \to X$ an alteration.