Lecture 10: Flattening stratifications, functoriality properties of Hilb and Quot

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1 Flattening stratifications (cont.)

Theorem 1. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \bigsqcup_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

We began last time by proving the following special case corresponding to $X = S$.

Proposition 1. *Let $\mathcal{F}$ be a coherent sheaf on S Noetherian. Then there exists a finite locally closed stratification $\{S_d\}$ of S such that $\mathcal{F}|_{S_d}$ is locally free of rank d. Moreover, for any locally Noetherian scheme T, a morphism $\varphi : T \to S$ factors as $T \to S_d \subset S$ if and only if $\varphi^*\mathcal{F}$ is locally free of rank d.*

Now the idea of the proof in general is to use the following result from lecture 5.

Corollary 1. *Let $f : X \to S$ be a projective morphism with S noetherian. If $\mathcal{F}$ is a coherent sheaf on X then $\mathcal{F}$ is flat over S if and only if $f_*\mathcal{F}(d)$ is a locally free sheaf of finite rank for all $d \gg 0$.*

The idea of the proof then is to apply the proposition to the sheaves $f_*\mathcal{F}(d)$ and find a stratification that is universal for these pushforwards being locally free. Then by the corollary and some base-change arguments, this stratification will be universal for $\mathcal{F}$ being flat.

Proof. (flattening stratifications in general)

The first step of the proof is to bound the Hilbert polynomials of the fibers of $\mathcal{F}$ using the generic freeness theorem (from problem set 1).

Theorem 2. *Let A be noetherian integral domain, B a finitely generated A-algebra, and M a finite B-module. Then there exists an $f \in A$ such that M_f is a free A_f-module.*

Now we will use this to produce a finite stratification of S into reduced locally closed subschemes $V_i \subset S$ for $i = 1, \dots, m$ such that $\mathcal{F}|_{V_i}$ is flat over V_i (via the pullback $X_{V_i} \to V_i$ of $X \to S$). Toward this end, we may take S_{red} and assume that S is reduced. Let $\cup Y_j = S$ be the decomposition into irreducible components and fix a component Y_0. Let $U_0 \subset Y_0$ be the complement of where Y_0 meets Y_i:

$$U_0 = Y_0 \setminus \{Y_i \cap Y_1\}_{i \neq 0}.$$

Now U_0 is an integral scheme. Let Spec $A \subset U_0$ be a dense open affine subscheme where A is an integral domain. Then we may apply the generic freeness theorem to the pullback $\mathcal{F}_A$ to $X_A \to$ Spec A. This gives us an open subscheme Spec $A_f \subset$ Spec $A \subset U_0$ such that $\mathcal{F}_{A_f}$ is flat over Spec A_f. Let $V_0 =$ Spec A_f and let $S_1 = S \setminus V_0$ its closed complement. Now we repeat in this way to produce an integral open subset $V_i \subset S_i$ where $\mathcal{F}_{V_i}$ is flat over V_i. By Noetherian induction, this process terminates so we get a stratification of S by locally closed subsets $\{V_0, \dots, V_m\}$ with the desired property.

Let us denote by $f_i : X_i \to V_i$ the pullback of $X \to S$ and by $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to $\mathcal{F}_i$. Now by construction, $\mathcal{F}_i$ is flat over V_i and by constancy of Hilbert polynomials, for each $s \in V_i$, the Hilbert polynomial $P_{\mathcal{F}_s}(d)$ is constant say equal to a polynomial P_m. Thus there are finitely many polynomials $\{P_1(d), \dots, P_m(d)\}$ such that for each $s \in S$, $P_{\mathcal{F}_s}(d) = P_m(d)$ for some m. Next, by Serre vanishing, there exists a d_i such that

$$R^j(f_i)_*\mathcal{F}_i(d) = 0$$

for all $d \geq d_i$. In this case, by cohomology and base change, $(f_i)_*\mathcal{F}_i(d)$ is locally free of rank $P_{\mathcal{F}_s}(d)$ and the basechange map

$$(f_i)_*\mathcal{F}_i(d) \otimes k(s) \to H^0(X_s, \mathcal{F}_s(d))$$

is an isomorphism. Letting $N = \max\{d_i\}$, we now have the following:

(1) There are finitely many polynomials $P_1, \dots, P_m$ such that for each $s \in S$, $P_{\mathcal{F}_s}(d) = P_i(d)$ for some i;

(2) $H^i(X_s, \mathcal{F}_s(d)) = 0$ for all $d \geq N$;

(3) $f_*\mathcal{F}(d) \otimes k(s) \cong H^0(X_s, \mathcal{F}_s(d))$ has dimension $P_i(d)$ for all $d \geq N$.

Now we will construct the flattening stratification for $\mathcal{F}$ using properties $(1), (2)$ and (3). Note that the preliminary stratification into reduced strata V_i above was just an auxillary tool to prove properties $(1), (2)$ and (3).

Fix n such that $\deg P_{\mathcal{F}_s}(d) \leq n$ for all $s \in S$ which exists by (1).[1] We have the following fact.

Fact 1. *Let Pol_n be the set of polynomials over $\mathbb{Q}$ of degree at most n. Then for any N,*

$$Pol_n \to \mathbb{Z}^{n+1} \tag{1}$$
$$P \mapsto (P(N), P(N+1), \dots, P(N+n)) \tag{2}$$

is a bijection.

[1] Actually we already knew this because $X \subset \mathbb{P}^n_S$ for some n and we can take this n. However, this is only because we are using a stronger version of projectivity in this class.

Now we can apply the flattening stratification in the special case of the coherent sheaves $\{\mathcal{E}_i := f_*\mathcal{F}(N+i)\}_{i=0}^n$ on S. Thus for each i and e, we have a stratum $W_{i,e}$ that is universal for the property that $\mathcal{E}_i$ is locally free of rank e. In particular, for any $s \in W_{i,e}$, by the base change properties (2) and (3), we have $e = \mathrm{rk}\mathcal{E}_i|_{W_{i,e}} = P_{\mathcal{F}_s}(N+i)$. Now for any sequence $(e_0, \ldots, e_n) \in \mathbb{Z}^{n+1}$, which by the basic fact corresponds to a polynomial P, we have the scheme theoretic intersection

$$W_P^0 := \bigcap_{i=0}^n W_{i,e_i}.$$

By definition, a map $\varphi : T \to S$ factors through W_P^0 if and only if $\varphi^* f_*\mathcal{F}(N+i)$ is locally free of rank $e_i = P(N+i)$ for $i = 0, \ldots, n$. In particular, $s \in W_P^0$ if and only if $P_{\mathcal{F}_s}(d) = P(d)$ and so by finiteness of the Hilbert polynomials, $\{W_P^0\}$ is a finite locally closed stratification of S which has the correct closed points. However, we need to determine the correct scheme structure.

By the vanishing condition (2), we know that the formation of $f_*\mathcal{F}(N+a)$ is compatible with arbitrary base change for all $a \geq 0$. Now for each $d \geq 0$, let we can apply the flattening stratification to the sheaf $f_*\mathcal{F}(N+n+d)|_{W_P^0}$ to obtain a locally closed subscheme W_P^d universal for this sheaf being locally free of rank $P(N+n+d)$. Note that at every closed point of W_P^0, the rank of $f_*\mathcal{F}(N+n+d)$ is equal to $P(N+n+d)$ and so W_P^d has the same underlying reduced subscheme. In particular, it is actually a closed subscheme of W_P^0 and so is cut out by some ideal I_P^d. Now consider the chain

$$I_P^1 \subset I_P^1 + I_P^2 \subset \ldots.$$

By the Noetherian condition, this sequence stabilizes to some ideal I cutting out a closed subscheme $S_P \subset W_P^0$ with the same underlying reduced scheme. Equivalently, S_P is the scheme theoretic intersection of the W_P^d for all d.

Since $S_P \subset W_P^0$ is a homeomorphism of underlying topological spaces, then $\{S_P\}$ is a finite locally closed stratification of S. By definition, $\varphi : T \to S$ factors through S_P if and only if for all $a \geq 0$,

$$\varphi^* f_*\mathcal{F}(N+a)$$

is locally free of rank $P(N+a)$ but by the base change property,

$$\varphi^* f_*\mathcal{F}(N+a) = (f_T)_*\mathcal{F}_T(N+a).$$

Therefore, $\varphi : T \to S$ factors through S_P if and only if $(f_T)_*\mathcal{F}_T(N+a)$ is locally free of rank $P(N+a)$ for all $a \geq 0$ if and only if $\mathcal{F}_T$ is flat over T with Hilbert polynomial $P(d)$ by the corollary from lecture 5. Thus S_P has the required universal property.

□

2 Functoriality properties of Hilbert schemes

2.1 Closed embeddings

We proved the following in Step 1 of the construction of Hilbert schemes but its useful enough to make explicit.

Proposition 2. *Let $i : X \to Y$ be a closed embedding of projective S-schemes for S noetherian. Then there is a natural closed embedding $i_* : \mathrm{Hilb}^P_{X/S} \to \mathrm{Hilb}^P_{Y/S}$.*

2.2 Base-change

Let $f : X \to S$ be a projective morphism to a noetherian scheme and $S' \to S$ any morphism. Consider the pullback

$$\begin{CD} X' @>>> X \\ @V{f'}VV @VV{f}V \\ S' @>>> S. \end{CD}$$

Note that f' is projective. Suppose $\mathcal{E}$ is any coherent sheaf on X and let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ to X'. The following is clear from the definition of the Quot functor.

Proposition 3. *The following is a pullback square of functors.*

$$\begin{CD} \mathcal{Q}_{\mathcal{E}',X'/S'} @>>> \mathcal{Q}_{\mathcal{E},X/S} \\ @VVV @VVV \\ S' @>>> S \end{CD}$$

In particular, we conclude that

$$\mathrm{Quot}_{\mathcal{E}',X'/S'} \cong \mathrm{Quot}_{\mathcal{E},X/S} \times_S S'$$

2.3 Pullbacks

Let $f : X \to Y$ be a flat morphism of projective S-schemes. Since flatness and the Hilbert polynomial are stable under base-change, we have the following:

Proposition 4. *There is a pullback morphism*

$$f^* : \mathrm{Hilb}_{Y/S} \to \mathrm{Hilb}_{X/S}$$

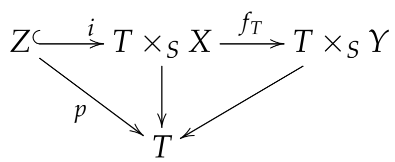
induced by taking a closed subscheme $Z \subset T \times_S Y$ to the pullback $g^{-1}(Z) \subset T \times_S X$.

Proof. Since $f : X \to Y$ is flat, then so is $f_T : T \times_S X \to T \times_S Y$ and so $g^{-1}(Z) \to Z$ is also flat. Then the composition $g^{-1}(Z) \to Z \to T$ is also flat and so is an element of $\mathrm{Hilb}^P_{X/S}(T)$. □

2.4 Pushforwards

Let $f : X \to Y$ be a morphism of projective S-schemes. We can ask if there is a "push-forward" map f_* in general. We saw above there is if f is a closed embedding.

Let us consider the diagram



where $p : Z \to T$ is an element of $\mathrm{Hilb}_{X/S}(T)$. The question is whether the composition $f_T \circ i : Z \to T \times_S Y$ is a closed embedding and thus gives an element of $\mathrm{Hilb}_{Y/S}(T)$ which we can use to define $f_*(p : Z \to T)$. Of course this won't be true in general but it turns out it holds on an open subscheme of $\mathrm{Hilb}_{X/S}$.

Theorem 3. *There exists a universal open subscheme* $\mathrm{Hilb}_{X\to Y/S} \subset \mathrm{Hilb}_{X/S}$ *on which* f_* *defined by* $Z \subset T \times_S X$ *maps to* $f_T \circ i : Z \to T \times_S Y$ *gives a morphism*

$$f_* : \mathrm{Hilb}_{X\to Y/S} \to \mathrm{Hilb}_Y.$$

Proof. Equivalently, we want to show there exists an open subscheme $U \subset \mathrm{Hilb}_{X/S}$ such that a morphism $\varphi : T \to \mathrm{Hilb}_{X/S}$ coresponding to a closed subscheme $i : Z \subset T \times_S X$ factors through U if and only if the composition $f_T \circ i : Z \to T \times_S Y$ is a closed embedding. Indeed if such a subscheme exists its clearly universal and then $f_T \circ i$ is an element of $\mathrm{Hilb}_{Y/S}$ such $Z \to T$ is flat and proper regardless of embedding. Now the existence of this $U = \mathrm{Hilb}_{X\to Y/S}$ follows from applying the next lemma to the universal map $g := f_{\mathrm{Hilb}_{X/S}} \circ i$.

$$\begin{array}{ccccc} \mathrm{Hilb}_{X/S} \times_S X & \xleftarrow{\;i\;} & \mathcal{Z} & \xrightarrow{\;g\;} & \mathrm{Hilb}_{X/S} \times_S Y \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Hilb}_{X/S} & & \end{array}$$

□

Lemma 1. *Let* $p : Z \to T$ *and* $q : Y \to T$ *be projective* T*-schemes with* p *flat and let* $g : Z \to T$ *a morphism. Then there exists an open subscheme* $U \subset T$ *such that* $\varphi : T' \to T$ *factors through* U *if and only if* $\varphi^* g : Z_{T'} \to Y_{T'}$ *is a closed embedding.*

Proof. Since Z and Y are projective, the morphism g is proper and so we may replace Y by the image of g and assume without loss of generality that Y is the scheme theoretic image of g. Then g is a closed embedding if and only if it is an isomorphism and so the result reduces to the lemma proved during our study of Hom-schemes.

□