Lecture 11: Weil restriction, quasi-projective schemes

10/16/2019

1 Weil restriction of scalars

Let $S' \to S$ be a morphism of schemes and $X \to S'$ an S'-scheme. The *Weil restriction of scalars* $R_{S'/S}(X)$, if it exists, is the S-scheme whose functor of pointsis given by

$$\mathrm{Hom}_S(T, R_{S'/S}(X)) = \mathrm{Hom}_{S'}(T \times_S S', X).$$

Classically, the restriction of scalars was studied in the case that $S' \to S$ is a finite extension of fields $k \subset k'$. In this case, $R_{S'/S}(X)$ is roughly given by taking the equations of X/k' and viewing that as equations over the smaller field k.

Theorem 1. *Let $f : S' \to S$ be a flat projective morphism over S Noetherian and let $g : X \to S'$ be a projective S'-scheme. Then the restriction of scalars $R_{S'/S}(X)$ exists and is isomorphic to the open subscheme $\mathrm{Hilb}^P_{X \to S'/S} \subset \mathrm{Hilb}^P_{X/S}$ where P is the Hilbert polynomial of $f : S' \to S$.*

Proof. Note that $\mathrm{Hilb}^P_{S'/S} = S$ with universal family given by $f : S' \to S$. Then on $\mathrm{Hilb}^P_{X \to S/S'}$ we have a well defined pushforward

$$g_* : \mathrm{Hilb}^P_{X \to S'/S} \to S$$

given by composing a closed embedding $i : Z \subset T \times_S X$ with $g_T : T \times_S X \to T \times_S S'$. On the other hand, since the Hilbert polynomials agree, then the closed embedding $g_T \circ i : Z \to T \times_S S'$ must is a fiberwise isomorphism and thus an isomorphism. Therefore, $g_T \circ i : Z \to T \times_S X = (T \times_S S') \times_{S'} X$ defines the graph of an S' morphism

$$T \times_S S' \to X.$$

This gives us a natural transformation

$$\mathrm{Hilb}^P_{X \to S'/S} \to \mathrm{Hom}_{S'}(- \times_S S', X). \tag{1}$$

On the other hand, a T-point of the right hand side, $\varphi \in \mathrm{Hom}_{S'}(T \times_S S', X)$, gives us a graph

$$\Gamma_\varphi \subset T \times_S S' \times_{S'} X$$

which maps isomorphically to $T \times_S S'$. Since $S' \to S$ is flat, so is $\Gamma_\varphi \to T$ and thus defines an element of $\mathrm{Hilb}^P_{X \to S'/S}(T)$ giving an inverse to (1).

□

2 Hilbert and Quot functors for quasi-projective schemes

Next, we will generalize Hilbert and Quot functors to quasi-projective morphisms $f : X \to S$. Given a coherent sheaf $\mathcal{E}$ on X, we define the Quot functor just as before.

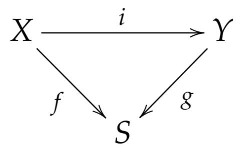
$$\mathcal{Q}_{\mathcal{E},X/S}(T) = \{q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ flat and proper over } T\}$$

Theorem 2. *Let $f : X \to S$ be a quasi-projective morphism over S noetherian and let $\mathcal{E}$ be a coherent sheaf on X. Then*

$$\mathcal{Q}_{\mathcal{E},X/S} = \bigsqcup \mathcal{Q}^P_{\mathcal{E},X/S}$$

over Hilbert polynomials P and each component $\mathcal{Q}^P_{\mathcal{E},X/S}$ is representable by a quasi-projective scheme over S.

Proof. Since $f : X \to S$ is quasi-projective, there is a projective $g : Y \to S$ and an open embedding $i : X \to Y$ such that the diagram



commutes.

Lemma 1. *There exists a coherent sheaf $\mathcal{E}'$ on Y such that $\mathcal{E}'|_X = \mathcal{E}$.*

Proof. Exercise. □

Given an element $(q, \mathcal{F}) \in \mathcal{Q}^P_{\mathcal{E},X/S}(T)$, we can consider the composition $q' : \mathcal{E}' \to \mathcal{E}|_X \to \mathcal{F}$, an object of $\mathcal{Q}^P_{\mathcal{E}',Y/S}(T)$. This gives us a natural transformation

$$\mathcal{Q}^P_{\mathcal{E},X/S} \to \mathcal{Q}^P_{\mathcal{E}',Y/S}.$$

We wish to show this is an open embedding. This follows from the following lemma.

Lemma 2. *Let $p : Y \to S$ be a proper morphism, $Z \subset Y$ a closed subscheme, and $\mathcal{F}$ a coherent sheaf on Y. Then there exists an open subscheme $U \subset S$ such that a morphism $\varphi : T \to S$ factors through U if and only if the support of the sheaf $\mathcal{F}_T$ on Y_T is disjoint from the closed subscheme Z_T.*

Proof. Exercise. □

Now we apply the lemma to $\mathcal{F}^P_{\mathcal{E}',Y/S}$ the universal sheaf on $p : \mathrm{Quot}^P_{\mathcal{E}',Y/S} \times_S Y \to \mathrm{Quot}^P_{\mathcal{E}',Y/S}$ with the closed subscheme Z being the complement of the open subscheme $\mathrm{Quot}^P_{\mathcal{E}',Y/S} \times_S X$. Then we get an open

$$U \subset \mathrm{Quot}^P_{\mathcal{E}',Y/S}$$

such that $\varphi : T \to \mathrm{Quot}^P_{\mathcal{E}',Y/S}$ factors through U if and only if the support of $\mathcal{F}_T$ lies in X_T. This is precisely the subfunctor $\mathcal{Q}^P_{\mathcal{E},X/S}$ so we conclude this subfunctor is representable by the quasi-projective scheme U. □

2.1 Hironaka's example

The Hilbert functor need not be representable by a scheme outside of the quasi-projective case. Indeed we have the following example due to Hironaka. For simplicity, we will work over the complex numbers.

Let X be a smooth projective 3-fold with two smooth curves C and D intersecting transversely in two points x and y. Consider the open subset $U_x = X \setminus x$ and let V_x be the variety obtained by first blowing up $C \setminus x$ inside U_x, then blowing up the strict transform of $D \setminus x$ inside the first blowup. Similarly, let $U_y = X \setminus y$ and let V_y be obtained by first blowing up $D \setminus y$ then blowing up the strict transform of $C \setminus y$.

Let π_x, π_y be the natural morphisms from the blowups to the open subsets of X. Then by construction $\pi_x^{-1}(U_x \cap U_y) \cong \pi_y^{-1}(U_x \cap U_y)$ so we can glue them together to obtain a variety Y with a morphism $\pi : Y \to X$.

Claim 1. *The variety Y is proper but not projective.*

Proof. The morphism π is proper by construction so Y is proper. Let l and m be the preimages of a general point on C and D respectively. These are algebraic equivalence classes of curves on Y. The preimage $\pi^{-1}(x)$ is a union of two curves l_x and m_x where $m \sim_{alg} m_x$ and $l \sim_{alg} l_x + m_x$. Similarly, $\pi^{-1}(y)$ is a union of l_y and m_y where $l \sim_{alg} l_y$ and $m \sim_{alg} l_y + m_y$. Putting this together, we get $l_x + m_y \sim_{alg} 0$. But l_x and m_y are irreducible curves so if Y is projective, it would have an ample line bundle which has positive degree on $l_x + m_y$, a contradiction.

□

Now we pick X, C and D such that X has a a fixed point free involution τ which sends C to D and x to y. Then τ lifts to an involution on Y. We will study quotients of varieties in more detail later, but for now we can consider the quotient Y/τ as a complex manifold.

Claim 2. *The quotient Y/τ is not an algebraic variety.*

Proof. Let l', m', l_0, and m_0 be the images under $Y \to Y/\tau$ of l, m, l_x and m_x respectively, viewed as homology classes. Note that l_y and m_y map to the same classes. Then the algebraic equivalences show the following equalities of classes in homology:

$$[m_0] = [m'] = [l'] = [m_0] + [l_0]$$

which implies that the homology class of l_0 vanishes.

Suppose Y/τ is a variety let $t \in l_0$ be a point. Let U be an affine open neighborhood of t in Y/τ. Pick an irreducible surface $S_0 \subset U$ passing through t but not containing $l_0 \cap U$, and S be the closure of S_0 in Y/τ. Then on the one hand, the intersection number $S \cap l_0 > 0$ since its the intersection of two irreducible subvarieties meeting at a finite number of points, but on the other hand $S \cap l_0 = 0$ since $[l_0] = 0$, a contradiction. □

Claim 3. *The Hilbert functor $\mathcal{H}_{Y/\mathbb{C}}$ is not representable by a scheme.*

Proof. Let $R \subset Y \times Y$ be the closed subset defined as

$$R = Y \times_{Y/\tau} Y.$$

We can consider the action of $G = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$ on Y as a morphism $m : G \times Y \to Y$. There is also a projection $p_Y : G \times Y \to Y$. Then the product of these two maps gives a proper morphism $G \times Y \to Y \times Y$ that is an isomorphism onto R. In particular, the projection $R \to Y$ is flat and proper and so $R \subset Y \times Y$ defines a flat family of closed subschemes of Y parametrized by Y, i.e., a morphism

$$Y \to \mathcal{H}_{Y/\mathbb{C}}.$$

Suppose the latter is representable by a scheme $\mathrm{Hilb}_{Y/\mathbb{C}}$. Since Y is proper so is $\mathrm{Hilb}_{Y/\mathbb{C}}$[1] Then $Y \to \mathrm{Hilb}_{Y/\mathbb{C}}$ is a proper morphism and so its image Z is a closed subscheme of $\mathrm{Hilb}_{Y/\mathbb{C}}$. On the other hand, the underlying map complex spaces $Y \to Z$ is exactly the quotient map $Y \to Y/\tau$ since the fibers of $R \to Y$ are exactly the orbits of τ and so $Y \to \mathrm{Hilb}_{Y/\mathbb{C}}$ sends a point to its orbit. This contradicts the fact that Y/τ is not a scheme.

□

[1]Note that the proof of properness was purely functorial and did not require representability or projectivity.