Lecture 12: The Picard functor

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1 Picard groups

Our goal now is to study the representability properties of the Picard functor. Recall the definition of the Picard group.

Definition 1. *Let X be a scheme. The Picard group* $\mathrm{Pic}(X)$ *is the set of line bundles (or invertible sheaves) on X with group operation given by tensor product.*

Recall the following well known fact.

Lemma 1. *There is a canonical isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

2 Picard functors

Let $f : X \to S$ be a scheme over S. We want to upgrade the Picard group to a functor on Sch_S. Since line bundles pull back to line bundles, we have a natural functor given by

$$T \mapsto \mathrm{Pic}(X_T).$$

Thus functor is the *absolute Picard functor*.

It is natural to ask if this functor is representable. It turns out this is not the case.

Claim 1. *The absolute Picard functor is not a sheaf.*

Proof. Let L be a line bundle on T such that f_T^*L is not trivial. Let $\{U_\alpha\}$ be an open cover of T that trivializes the bundle L. Then the pullback of f_T^*L to X_{U_α} is trivial and so L is in the kernel of the map

$$\mathrm{Pic}(X_T) \to \mathrm{Pic}\left(\bigsqcup_\alpha X_{U_\alpha}\right) = \prod_\alpha \mathrm{Pic}(X_{U_\alpha}).$$

□

Since the problem is the line bundles pulled back from the base scheme T, one proposed way to fix this is the following definition of the relative Picard functor.

Definition 2. *The relative Picard functor* $\mathrm{Pic}_{X/S} : Sch_S \to Set$ *is given by*

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/f_T^*\mathrm{Pic}(T)$$

where $f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X)$ *is the pullback map.*

The representability of $\mathrm{Pic}_{X/S}$ is still a subtle question and even the sheaf property is subtle and doesn't always hold. However, it does under some assumptions.

Definition 3. *We call a proper morphism $f : X \to S$ an algebraic fiber space if the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism. We say it is a universal algebraic fiber space if for all $T \to S$, the natural morphism $\mathcal{O}_T \to (f_T)_*\mathcal{O}_{X_T}$ is an isomorphism.* [1]

Proposition 1. *Suppose $f : X \to S$ is a universal algebraic fiber space and that there exists a section $\sigma : S \to X$.*[2] *Then $\mathrm{Pic}_{X/S}$ is a Zariski sheaf.*

Proof. Let us consider the Zariski sheafification $\mathrm{Pic}_{X/S,\ Zar}$ of $\mathrm{Pic}_{X/S}$. This is the sheafification of the functor

$$T \mapsto H^1(X_T, \mathcal{O}^*_{X_T}).$$

If we restrict this to the category of open subschemes of a fixed T, what we get is the sheaf

$$R^1(f_T)_*\mathcal{O}^*_{X_T}.$$ [3]

Thus $\mathrm{Pic}_{X/S,\ Zar}(T)$ is the global sections of the sheaf $R^1(f_T)_*\mathcal{O}^*_{X_T}$ on T:

$$\mathrm{Pic}_{X/S,\ Zar}(T) = H^0(T, R^1(f_T)_*\mathcal{O}^*_{X_T}).$$

Now consider the Leray spectral sequence

$$H^j(T, R^i(f_T)_*\mathcal{O}^*_{X_T}) \implies H^{i+j}(X_T, \mathcal{O}^*_{X_T}).$$

There is a 5-term exact sequence associated to any spectral sequence which in this case is given by

$$\begin{aligned} 0 \to H^1(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^1(X, \mathcal{O}^*_{X_T}) \to H^0(T, R^1(f_T)_*\mathcal{O}^*_{X_T}) \\ \to H^2(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^2(X, \mathcal{O}^*_{X_T}). \end{aligned}$$

Since f is a universal algebraic fiber space, $(f_T)_*\mathcal{O}^*_{X_T} \cong \mathcal{O}^*_T$ and so the first map in the exact sequence can be identified with the pullback

$$f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X_T).$$

Thus we have an exact sequence

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S,\ Zar}(T).$$

We want to show that the last map above is surjective so that

$$\mathrm{Pic}_{X/S,\ Zar}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T) = \mathrm{Pic}_{X/S}(T).$$

By exactness, it suffices to show that

$$H^2(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^2(X, \mathcal{O}^*_{X_T})$$

[1]Some sources in the literature require algebraic fiber spaces to be projective not just proper morphisms.

[2]That is, σ is a morphism with $f \circ \sigma = id_S$.

[3]Some take this as the definition of the higher direct image functors and then you have to prove their properties, otherwise you can define the higher direct image functors as derived functors and check they agree with this sheafification.

is injective. This map is given by pulling back by f_T. Since f_T has a section given by σ_T, then we have

$$\sigma_T^* \circ f_T^* = id : H^2(T, (f_T)_* \mathcal{O}_{X_T}^*) \to H^2(T, (f_T)_* \mathcal{O}_{X_T}^*).$$

Therefore f_T^* is injective. $\square$

Remark 1. *In the simplest case when $S = \text{Spec } k$ is the spectrum of a field, the condition that $f : X \to S$ is a unviersal algebraic fiber space is equivalent to X being geometrically connected and geometrically reduced. The condition that there exist a section $\sigma : S \to X$ is exactly saying that X has a k-rational point. Note that this is always true after a separable field extension of k, that is, it holds étale locally. This suggests that to study the relative Picard functor in greater generality, one should consider the sheafification of $\text{Pic}_{X/S}$ in the étale topology. Indeed it turns out that in the most general setting one should consider the fppf*[4] *sheafification of $\text{Pic}_{X/S}$. In the setting above where there exists a section, $\text{Pic}_{X/S}$ is already an étale and even fppf sheaf. To avoid getting into details of Grothendieck topologies and descent theory at this point in the class, we will stick with the case where a section σ exists.*

3 Some remarks and examples

Note that the relative Picard functor has the same k-points as the absolute Picard functor:

$$\text{Pic}_X(k) = \text{Pic}(X_k) = \text{Pic}(X_k)/\text{Pic}(\text{Spec } k) = \text{Pic}_{X/S}(k).$$

Thus the points of the relative Picard scheme of $f : X \to S$, if it exists, can be identified with line bundles on the fibers X_s of f. The difference between Pic_X and $\text{Pic}_{X/S}$ is only in how we glue together fiberwise line bundles into line bundles on the total space X.

Even when $\text{Pic}_{X/S}$ is a Zariski sheaf, it may still exhibit some pathologies.

Example 1. *(The Picard functor is not separated.) Let*

$$X = \{tf(x, y, z) - xyz = 0\} \subset \mathbb{P}^3_{\mathbb{A}^1_t}$$

be a family of cubic curves in the plane over $k = \bar{k}$ an algebraically closed field, where $f(x, y, z)$ is a generic cubic polynomial so that the generic fiber of the projection $f : X \to \mathbb{A}^1_t = S$ is smooth and irreducible. Note that f is a universal algebraic fuber space and we can pick $f(x, y, z)$ appropriately so that a section σ exists. The special fiber at $t = 0$, given by $V(xyz)$, is the union of three lines l_1, l_2 and l_3. We will show that in this case, $\text{Pic}_{X/S}$ fails the valuative criteria for the property of being separated.

Indeed suppose L^0 is a line bundle on $X \setminus X_0$ viewed as an element of $\text{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ and suppose there exists some line bundle L on X such that $L|_{X \setminus X_0}$ gives the same element of $\text{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ as L^0. Explicitly, this means that there exists some line bundle G on $\mathbb{A}^1 \setminus 0$ such that

$$L|_{X \setminus X_0} \otimes f^* G = L^0.$$

Then we claim that the twist $L(l_i) = L \otimes \mathcal{O}_X(l_i)$ by a component l_i of the central fiber gives another element of $\text{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ extending L^0 that is not equal to L. In particular, the map $\text{Pic}_{X/S}(\mathbb{A}^1) \to \text{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ is not injective and so the valuative criterion fails.

[4] faithfully flat and of finite presentation

To verify the claim, note that $\mathcal{O}_X(l_i)|_{X\setminus X_0} \cong \mathcal{O}_{X_0}$ and so $L(l_i)$ does indeed give an extension of L^0 in $\mathrm{Pic}_{X/S}$. On the other hand, L and $L(l_i)$ give the same element of $\mathrm{Pic}_{X/S}(\mathbb{A}^1)$ if and only of $\mathcal{O}_X(l_i)$ is pulled back from $\mathbb{A}^1$ which does not hold since the restriction to the scheme theoretic fiber $\mathcal{O}_X(l_i)|_{f^{-1}}(\{t=0\})$ is a nontrivial line bundle.

This example shows that to get a well behaved relative Picard functor, we should restrict to the case that the fibers of $f : X \to S$ are integral. Indeed if the fibers are integral, then any fiber component $\mathcal{O}_X(F)$ that we can twist by is pulled back from the base and so this problem doesn't occur.

Example 2. *(The Picard functor need not be universally closed.) Let $X = \{y^2z - x^2(x-z) = 0\}$ in $\mathbb{P}^3_k$ with $S = \mathrm{Spec}\, k$ for $k = \bar{k}$. Then $f : X \to \mathrm{Spec}\, k$ is a universal algebraic fiber space and it has a section given by the rational point $[0,1,0]$. Consider the subscheme $D \subset \mathbb{A}^1 \times_k X$ given by the graph of the morphism*

$$\varphi : \mathbb{A}^1 \to X, \quad t \mapsto [t^2+1, t(t^2+1), 1].$$

Let $t_\pm = \pm\sqrt{(-1)}$, $U = \mathbb{A}^1 \setminus \{t_+, t_-\}$, and $X_U = X \setminus \{X_{t_+} \cup X_{t_-}\}$. Then $D|_U \subset X_U$ is contained in the regular locus of $X_U = U \times X$ and so the ideal sheaf of $D|_U$ is a line bundle denoted $\mathcal{O}_{X_U}(-D|_U)$. If an extension of $\mathcal{O}_{X_U}(-D|_U)$ to $\mathbb{A}^1$ exists as an element of the relative Picard functor, then it must be represented by a line bundle on X, and in particular, it must be flat over $\mathbb{A}^1$. On the other hand, we know that the ideal sheaf $\mathcal{I}_D$ is a flat extension of $\mathcal{O}_{X_U}(-D|_U)$. One can check that if the extension of $\mathcal{O}_{X_U}(-D|_U)$ exists as an element of the relative Picard functor, it must be equal to $\mathcal{I}_D$ up to twisting by a line bundle on the base. Since $D \to \mathbb{A}^1$ is flat, $\mathcal{I}_D|_{t_\pm} = \mathcal{I}_{D_{t_\pm}}$ but $D_{t_\pm}$ is the closed point $[0,0,1]$. The completed local ring of $X_{t_\pm}$ at this point is given by $k[\![x,y]\!]/(xy)$ and it has maximal ideal (x,y) corresponding to the point $[0,0,1]$. It is easy to see that (x,y) is not *a free $k[\![x,y]\!]/(xy)$-module and thus $\mathcal{I}_D$ is not a line bundle and so no extension of $\mathcal{O}_{X_U}(-D|_U)$ as an element of the relative Picard functor can exist.*

In the above example, what goes wrong is that the flat limit of the given family of line bundles is not a line bundle, but rather the rank 1 torsion free sheaf $\mathcal{I}_{D_{t_\pm}}$. Thus suggests that at least in the case of an integral curve, one can compactify the Picard functor by allowing such sheaves. We will study this *compactified Picard scheme* in the case of integral curves lying on a smooth surface[5] later in the class.

4 Outline of the proof of the representability theorem

Our goal will be to prove the following theorem.

Theorem 1. *Let $f : X \to S$ be a flat projective morphism with integral fibers such that f is a universal algebraic fiber space and suppose there exists a section $\sigma : S \to X$.[6] Then the relative Picard functor $\mathrm{Pic}_{X/S}$ is a representable by a locally of finite type, separated S-scheme with quasi-projective connected components.*

The proof roughly proceeds in the following steps:

[5]so-called locally planar curves

[6]In fact all we will need is that $\mathrm{Pic}_{X/S}$ is a sheaf.

(I) Given a Cartier divisor, that is, a codimension one closed subschemes $D \subset X$ with locally free ideal sheaf $I_D \subset \mathcal{O}_X$, we can dualize to obtain a line bundle $L = I_D^{-1}$ with section $s : \mathcal{O}_X \to L$. This gives a set theoretic bijection

$$\{(L, s) \mid s : \mathcal{O}_X \to L \text{ is injective}\} \leftrightarrow \{\text{Cartier divisors}\}.$$

(II) We define a relative notion of Cartier divisors and prove that the moduli functor $CDiv_{X/S}$ of relative Cartier divisors is representable by an open subscheme of the Hilbert scheme $\mathrm{Hilb}_{X/S}$. In particular, we have a disjoint union

$$CDiv_{X/S} = \bigsqcup_P CDiv^P_{X/S}$$

over Hilbert polynomials where each component is quasi-projective.

(III) Using the bijection in (I), we construct a morphism of functors

$$CDiv^P_{X/S} \to \mathrm{Pic}^{P_1}_{X/S}$$

which on k-points is given by $(D \subset X_k) \mapsto L = I_D^{-1}$. Here P_1 is the Hilbert polynomial of I_D^{-1} which depends only on the Hilbert polynomial P of D and that of $f : X \to S$. Note that since f is flat, any line bundle on X_T is flat over T for any $T \to S$ and so there is a disjoint union

$$\mathrm{Pic}_{X/S} = \bigsqcup_P \mathrm{Pic}^P_{X/S}.$$

(IV) For a suitable choice of P and P_1, after twisting by a large enough multiple of $\mathcal{O}_X(1)$, the map

$$CDiv^P_{X/S} \to \mathrm{Pic}^{P_1}_{X/S}$$

is the quotient of $CDiv^P_{X/S}$ in the category of sheaves by a proper and smooth (and in particular flat) equivalence relation.

(V) We will study proper and flat equivalence relations and show that the quotient of a quasi-projective scheme by such equivalence relation exists as a quasi-projective scheme. This uses the existence of Hilbert schemes.