Lecture 13: Relative effective Cartier divisors

10/21/2019

1 The universal line bundle on $\mathrm{Pic}_{X/S}$

Recall last time we defined for an S-scheme $f : X \to S$ the relative Picard functor

$$\mathrm{Pic}_{X/S} : Sch_S \to Set \quad T \mapsto \mathrm{coker}(\mathrm{Pic}(T) \to \mathrm{Pic}(X_T)).$$

Under the assumption that f is a universal algebraic fiber space[1] and f admits a section $\sigma : S \to X$, we showed that $\mathrm{Pic}_{X/S}$ is a Zariski sheaf.

Our main goal will be to show the following:

Theorem 1. *Let $f : X \to S$ be a flat projecitve scheme over S Noetherian. Suppose S is a universal algebraic fiber space and admits a section $\sigma : S \to X$ and that the fibers of f are geometrically integral. Then $\mathrm{Pic}_{X/S}$ is representable by a locally of finite type scheme over S with quasi-projective connected components.*

Note that the elements of $\mathrm{Pic}_{X/S}(T)$ are not line bundles, but rather equivalence classes of line bundles under the equivalence given by tensoring by line bundles from the base T. In particular, even if the relative Picard functor is representable, it is not immediate that there exists an actual line bundle on $\mathrm{Pic}_{X/S} \times_S X$ that pulls back to the appropriate class in $\mathrm{Pic}_{X/S}(T)$ for all T. To show this, let us introduce the following variant of the relative Picard functor.

Definition 1. *Let $f : X \to S$ be a universal algebraic fiber space with section $\sigma : S \to X$. The σ-rigidified Picard functor is the functor*

$$\mathrm{Pic}_{X/S,\sigma} : Sch_S \to Set$$

such that

$$\mathrm{Pic}_{X/S,\sigma}(T) = \{(L, \alpha) \mid L \text{ is a line bundle on } X_T,\ \alpha : \mathcal{O}_T \to \sigma_T^* L \text{ is an isomorphism}\}/\sim$$

where $(L, \alpha) \sim (L', \alpha')$ if and only if there exists an isomorphism $\epsilon : L \to L'$ such that $\sigma_T^*\epsilon \circ \alpha = \alpha'$. $\mathrm{Pic}_{X/S,\sigma}$ is made into a functor by pullback.

Remark 1. *Using the σ-rigidification and the assumptions on f one can check directly that $\mathrm{Pic}_{X/S,\sigma}$ is a sheaf in the Zariski topology. In fact under these assumptions it is even a sheaf in the fppf topology.*

[1] For any $T \to S$, $(f_T)_*\mathcal{O}_{X_T} = \mathcal{O}_T$. Note this holds in particular if f is projective and the fibers of are geometrically integral.

Proposition 1. *Suppose $f : X \to S$ is a universal algebraic fiber space with section σ. Then $\mathrm{Pic}_{X/S,\sigma} \cong \mathrm{Pic}_{X/S}$ as functors.*

Proof. There is a natural transformation

$$\mathrm{Pic}_{X/S,\sigma} \to \mathrm{Pic}_{X/S}$$

given by forgetting the data of α and composing with the projection $\mathrm{Pic}_X \to \mathrm{Pic}_{X/S}$ from the absolute Picard functor. On the other hand, given an element $\mathrm{Pic}_{X/S}(T)$ represented by some line bundle L on X_T, the line bundle

$$L \otimes (f_T)^* \sigma^* L^{-1}$$

has a canonical rigidification given by the inverse of the isomorphism

$$\sigma^* L \otimes \sigma^* L^{-1} \to \mathcal{O}_T$$

and this gives an inverse

$$\mathrm{Pic}_{X/S}(T) \to \mathrm{Pic}_{X/S,\sigma}.$$

□

Corollary 1. *Suppose $f : X \to S$ is a universal algebraic fiber space with section $\sigma : S \to X$. Assume that the relative Picard functor is representable. Then there exists a $\sigma_{\mathrm{Pic}_{X/S}}$-rigidified line bundle $\mathcal{P}$ on $X \times_S \mathrm{Pic}_{X/S}$ that is universal in the following sense. For any S-scheme T and any line bundle L on X_T, let $\varphi_L : T \to \mathrm{Pic}_{X/S}$ be the corresponding morphism. Then $\varphi_L^* \mathcal{P}$ is σ_T-rigidified and*

$$L \cong \varphi_L^* \mathcal{P} \otimes f_T^* M$$

for some line bundle M on T. In particular, if $T = \mathrm{Spec}\, k$, then for any k-point $[L] \in \mathrm{Pic}_{X/S}(k)$, $\mathcal{P}|_{X_k} \cong L$.

2 Relative Cartier divisors

Recall that an effective Cartier divisor $D \subset X$ is a closed subscheme such that at each point $x \in D$, $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ where $f_x \in \mathcal{O}_{X,x}$ is a regular element. That is, D is a pure codimension one locally principal subscheme. Then the ideal sheaf of D is a line bundle $\mathcal{O}_X(-D)$ and the inclusion $\mathcal{O}_X(-D) \cong \mathcal{I}_D \hookrightarrow \mathcal{O}_X$ induces a section

$$s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$$

of the dual line bundle $\mathcal{O}_X(D)$ which is everywhere injective.

Definition 2. *Let L be a line bundle. A section $s \in H^0(X, L)$ is regular if $s : \mathcal{O}_X \to L$ is injective. Two pairs (s, L) and (s', L') of line bundles with regular sections are said to be equivalent if there exists an pair (α, t) where*

$$\alpha : L \to L'$$

is an isomorphism of line bundles and $t \in H^0(X, \mathcal{O}_X^*)$ is an invertible function such that $\alpha(a) = ts'$.

Given a line bundle and a regular section (s, L), the vanishing locus $V(s)$ is an effective Cartier divisor with ideal sheaf $s^\vee : L^{-1} \hookrightarrow \mathcal{O}_X$ and in this way we have a bijection

$$\{\text{effective Cartier divisors}\} \leftrightarrow \{(s, L) \mid s \text{ is a regular section}\}/\sim$$

where $\sim$ is the equivalence relation on pairs (s, L) given above. We wish to consider the relative notion.

Definition 3. *Let $f : X \to S$ be a morphism of schemes. A relative effective Cartier divisor is an effective Cartier divisor $D \subset X$ such that the projection $D \to X$ is flat.*

We will show that this notion is well behaved under base-change by any $S' \to S$.

Lemma 1. *Suppose $D \subset X$ is a relative effective Cartier divisor for $f : X \to S$. For any $S' \to S$, denote by $f' : X' \to S'$ the pullback. Then $D' = S' \times_S D \subset X'$ is a relative effective Cartier divisor for f'.*

Proof. Flatness of $D' \to S'$ is clear. We need to check that D' is cut out at each local ring $\mathcal{O}_{X',x'}$ by a regular element. Let x be the image of x' and consider the exact sequence

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_{D,x} \to 0$$

where the first map is multiplication by the regular element f_x. Pulling back along $S' \to S$ gives us a sequence

$$0 \to \mathcal{O}_{X',x'} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{D',x'} \to 0$$

which is exact since $\mathcal{O}_{D,x}$ is flat so the Tor term on the left vanishes. The first map is multiplication by f'_x, the pullback of f_x. Since it is injective, f'_x is a regular element.

□

Corollary 2. *Let $f : X \to S$ be a flat morphism and $D \subset X$ a subscheme flat over S. The following are equivalent:*

(a) D is a relative effective Cartier divisor;

(b) $D_s \subset X_s$ is an effective Cartier divisor for each $s \in S$.

Proof. (a) $\implies$ (b) by the previous lemma. Suppose (b) holds. We need to show that for all $x \in X$, $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ where f_x is a regular element. By (b), we have that $\mathcal{O}_{D,x} \otimes k(s) = \mathcal{O}_{X,x} \otimes k(s)/\bar{f}_x$ where $\bar{f}_x$ is a regular element of $\mathcal{O}_{X,x} \otimes k(s) = \mathcal{O}_{X_s,x}$. Now by Nakayama's lemma we can lift this to an generator f_x of $\mathcal{I}_D$ so that $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ and f_x a regular element. □

Now we can define the functor

$$CDiv_{X/S} : Sch_S \to Set$$

given by

$$CDiv_{X/S}(T) = \{\text{relative effective Cartier divisors } D \subset X_T\}$$

Proposition 2. *Let $f : X \to S$ be a flat and projective morphism over a Noetherian scheme S. Then $CDiv_{X/S}$ is representable by an open subscheme of $\mathrm{Hilb}_{X/S}$. If moreover f is a smooth morphism, then $CDiv_{X/S}$ is proper over S.*

Proof. Since an element $CDiv_{X/S}(T)$ is a closed subscheme $D \subset X_T$ which is flat and proper over T, $CDiv_{X/S}$ is a subfunctor of $\text{Hilb}_{X/S}$. We need to show that the inclusion $CDiv_{X/S} \to \text{Hilb}_{X/S}$ is an open subfunctor.

That is, suppose $D \subset X_T$ is flat and proper over T. We need to show there exists an open subset $U \subset T$ such that $\varphi : T' \to T$ factors through U if and only if $D_{T'} \subset X_{T'}$ is an effective Cartier divisor which by the previous lemma is equivalent to the the requirement that $D_t \subset X_t$ is an effective Cartier divisor for each $t \in T'$.

Toward this end, let H be the union of irreducible components of $\text{Hilb}_{X/S}$ which contain the image of $CDiv_{X/S}$ and let $D \subset X \times_S H = X_H$ be the universal proper flat cloesed subscheme over H. Note that the non-Cartier locus of $D \subset X_H$ is exactly the locus where I_D is not locally free of rank 1. Since X_H is locally Noetherian and I_D is coherent, the locus where I_D is locally free of rank 1 is locally closed by the locally free stratification (special case of flattening). On the other hand, for any point $x \in X_H \setminus D$, $\mathcal{I}_{D,x} = \mathcal{O}_{X,x}$ is free of rank 1 and thus the stratum contains a dense open subscheme of X_H.[2] Therefore this stratum is in fact open. Let $Z \subset X_H$ be its complement so that $x \in Z$ if and only if D is not Cartier at $x \in X$.

Now we let

$$U := H \setminus f_H(Z) \subset H.$$

Then U is open since f_H is proper and $t \in U$ if and only if for all $x \in X_t$, D is Cartier at x if and only if $D_t \subset X_t$ is an effective Cartier divisor (by the prevoius lemma). Then a T-point of H factors through U if and only if for all $t \in T$, $D_t \subset X_t$ is an effective Cartier divisor if and only if $D_T \subset X$ is an effective Cartier divisor so U represents the subfunctor $CDiv_{X/S}$.

Suppose now that f is smooth. We will use the valuative criterion. Let $T = \text{Spec } R$ be the spectrum of a DVR with generic point $\eta = \text{Spec } K$ and closed point $0 \in T$ and let D_η be an η point of $CDiv_{X/S}$. By properness of the Hilbert functor, we know there exists a unique $D \subset \text{Hilb}_{X/S}(T)$ such that $D|_\eta = D_\eta$. We need to check that $D \subset X_T$ is in fact a relative effective Cartier divisor. This is equivalent to $D_0 \subset X_0$ being Cartier. By flatness over a DVR, the subscheme D has no embedded points and is pure of codimension 1 since D_η is pure of codimension 1. Thus $D_0 \subset X_0$ is a pure codimension 1 subscheme with no embedded points. Since X_0 is smooth, the local rings are UFDs and by a fact of commutative algebra, height 1 primes on UFDs are principal and thus $I_{D_0,x}$ is a principal ideal of $\mathcal{O}_{X_0,x}$ generated by a regular element for each $x \in X_0$.

□

Example 1. *(A non-proper space of effective Cartier divisors) Let $X \subset \mathbb{P}^3_{\mathbb{A}^1_t}$ be defined by the following equation.*

$$t(xw - yz) + x^2 - yz = 0$$

For $t \neq 0$, this is the smooth quadric surface which has a family of lines defined by the ideal (x, y). Since X_t is smooth this is a Cartier divisor. However, over $t = 0$, X_0 is a singular quadric cone $x^2 - yz$ and one can check that the deal (x, y) is not locally principal at the point $[0, 0, 0, 1]$. Therefore this family of lines gives an element in $CDiv_{X/\mathbb{A}^1}(\mathbb{A}^1 \setminus 0)$ which does not extend to $CDiv_{X/\mathbb{A}^1}(\mathbb{A}^1)$.

[2]Why is $X_H \setminus D$ dense in X_H? This is clear if we add the assumption that the fibers of $f : X \to S$ are integral. In general I want to use the fact that H is the union of components with Hilbert polynomial equal to that of a Cartier divisor to show that $X_H \setminus D$ is dense inside each irreducible component of each fiber. For our purposes we can assume the fibers of $f : X \to S$ are integral since that is the only case we will consider when constructing Picard schemes.