Lecture 14: The Abel-Jacobi map

10/23/2019

1 Representable morphisms

Throughout the course we have used the notion of a subfunctor being an open or closed subfunctor. This is a special case of the notion of representable morphism between functors.

Definition 1. *Let F, G be two functors $Sch_S \to Set$. A morphism $F \to G$ between functors is said to be representable by schemes if for any scheme T and any morphism $T \to G$, the pullback of functors $T \times_G F$ is representable by a scheme.*

Using Yoneda's lemma, the morphism $F \to G$ is representable by schemes if and only if the following condition holds. Given an element $\xi \in G(T)$, there exists a scheme T' and a morphism $T' \to T$ such that for any scheme T'' and any morphism $\varphi : T'' \to T$, φ factors through T' if and only of $\xi_{T''} \in G(T'')$ is the image of an element $\zeta \in F(T'')$. From this it is clear that open and closed subfunctors are representable morphisms. More generally, we can define the following properties for representable morphisms.

Definition 2. *For any of the following properties, we say that a morphism of functors $F \to G$ is representable by $\mathcal{P}$ if and only if $F \to G$ is representable by schemes and for all $T \to G$, the morphism $T \times_G F \to T$ has property $\mathcal{P}$:*

(a) $\mathcal{P}$ = "closed embeddings",

(b) $\mathcal{P}$ = "open embeddings",

(c) $\mathcal{P}$ = "affine morphisms",

(d) $\mathcal{P}$ = "projective bundles"[1],

(e) $\mathcal{P}$ = "proper morphisms",

(f) $\mathcal{P}$ = "flat morphisms",

(g) $\mathcal{P}$ = "smooth morphisms",

(h) $\mathcal{P}$ = "finite morphisms",

(i) $\mathcal{P}$ = "étale morphisms".

Remark 1. *In fact this definition makes sense for any property $\mathcal{P}$ of morphisms such that (1) $\mathcal{P}$ is stable under base change, and (2) the property can be checked for a morphism of schemes after taking a Zariski open cover of the target[2]. More generally, one can work with algebraic spaces or algebraic*

[1] Recall that a projective bundle on a scheme S is an S-scheme of the form $\mathbb{P}(\mathcal{E})$ for $\mathcal{E}$ a coherent sheaf on S.
[2] That is, the property is local on the target.

stacks and then one can ask for morphisms of (pseudo)functors[3] *to be representable by schemes, or by algebraic spaces, or by algebraic stacks. Then it makes sense for such morphisms to be representable by $\mathcal{P}$ for any property that is fppf or fpqc local on the target.*

2 The Abel-Jacobi map

Recall last time we constructed the moduli space $CDiv_{X/S}$ of relative effective Cartier divisors for any flat and projective $f : X \to S$ over a Noetherian scheme S. The elements of $CDiv_{X/S}(T)$ are Cartier divisors $D \subset X_T$ which are flat over T. Then the ideal sheaf $\mathcal{O}_{X_T}(-D)$ of D is a line bundle on X_T with dual $\mathcal{O}_{X_T}(D)$. We wish to show that sending D to $\mathcal{O}_{X_T}(D)$ defines a natural transformation of functors $CDiv_{X/S} \to \mathrm{Pic}_{X/S}$.

Proposition 1. *The natural map*

$$CDiv_{X/S}(T) \to \mathrm{Pic}_{X/S}(T)$$

given by $(D \subset X_T) \mapsto [\mathcal{O}_{X_T}(D)]$ is a natural transformation of functors $CDiv_{X/S} \to \mathrm{Pic}_{X/S}$.

Proof. We need to check this map is functorial in T. Each side is made into a functor under pullback so concretely, we need to check that for any $\varphi : T' \to T$, $\mathcal{O}_{X_{T'}}(D_{T'}) = \varphi^*\mathcal{O}_{X_T}(D)$ up to twisting by a line bundle pulled back from T'. Now $\mathcal{O}_D$ is flat over T, so pulling back the ideal sequence

$$0 \to \mathcal{O}_{X_T}(-D) \to \mathcal{O}_{X_T} \to \mathcal{O}_D \to 0$$

gives

$$0 \to \varphi^*\mathcal{O}_{X_T}(D) \to \mathcal{O}_{X_{T'}} \to \mathcal{O}_{D_{T'}} \to 0$$

so in particular, $\varphi^*\mathcal{O}_{X_T}(-D) = \mathcal{O}_{X_{T'}}(-D_{T'})$. Now since $\mathcal{O}_{X_T}(-D)$ is locally free, we have

$$\varphi^*\mathcal{O}_{X_T}(D) = \varphi^*\mathcal{O}_{X_T}(-D)^{-1} = \mathcal{O}_{X_{T'}}(-D_{T'})^{-1} = \mathcal{O}_{X_{T'}}(D_{T'})$$

as required. □

This map is often called the Abel-Jacobi map and is denoted by

$$AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}.$$

Let us study the fibers of $AJ_{X/S}$ over a k-point. A k-point $t \in \mathrm{Pic}_{X/S}(k)$ for $\mathrm{Spec}\, k \to S$ a point of S corresponds to a line bundle L on X_k. Then by the bijection between Cartier divisors and line bundles with regular sections up to isomorphism, the fiber $AJ^{-1}_{X/S}(t)$ is the set of pairs (s, L) where $s : \mathcal{O}_X \hookrightarrow L$ is a regular section up to scaling:

$$AJ^{-1}_{X/S}(t) = H^0(X_k, L)^{reg}/H^0(X_k, \mathcal{O}^*_{X_k}).$$

If X_k is geometrically integral, then every nonzero section is regular and we have $AJ^{-1}_{X/S}(t) = \mathbb{P}(H^0(X_k, L))$ is a projective space. This observation is generalized by the following theorem.

[3] Functors to the category of groupoids rather than to sets. We will discuss this in more detail later.

Theorem 1. *Suppose $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Suppose further that the fibers of f are geometrically integral. Then the Abel-Jacobi map $AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$ is representable by a projective bundle. More precisely, for any scheme T and T-point $\varphi_L : T \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X_T, there exists a coherent sheaf $\mathcal{E}$ on T such that the pullback $AJ^{-1}_{X/S}(T) \to T$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$ over T. Moreover, if $R^1(f_T)_*L = 0$, then (1) $(f_T)_*L$ commutes with base change, (2) $\mathcal{E}$ and $(f_T)_*L$ are locally free and dual to each other, and (3) the formation of $\mathcal{E}$ commutes with base change. In particular, if $R^1(f_T)_*L = 0$ for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms.*

To prove this theorem, we will use the following proposition, which is on problem set 2, and follows from the existence of the Grothendieck complex.

Proposition 2. *Let $f : X \to S$ be a proper morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X which is flat over S. Then there exists a coherent sheaf $\mathcal{Q}$ on S with a functorial isomorphism*

$$\theta_{\mathcal{G}} : f_*(\mathcal{F} \otimes f^*\mathcal{G}) \to \mathcal{H}om_S(\mathcal{Q}, \mathcal{G}).$$

Corollary 1. *Suppose $R^1 f_*\mathcal{F} = 0$ so that $f_*\mathcal{F}$ is locally free and commutes with base change by cohomology and base change. Then $\mathcal{Q}$ is locally free and is dual to $f_*\mathcal{F}$. In particular, the formation of $\mathcal{Q}$ commutes with base change.*

Proof. Apply the proposition to the special case $\mathcal{G} = \mathcal{O}_S$ and use that dualizing commutes with tensor products for locally free modules. □

Remark 2. *Note that by definition, $\mathrm{Pic}_{X/S}$ is compatible with base change in the following sense. For any $S' \to S$ such that $f' : X' \to S'$ is the pullback of $f : X \to S$,*

$$\mathrm{Pic}_{X'/S'} = \mathrm{Pic}_{X/S} \times_S S'$$

as functors. Moreover, if f satisfies any of the above assumptions then so does f'. Note also that $CDiv_{X/S} \times_S S' = CDiv_{X'/S'}$. Indeed we already discussed the Hilbert schemes have this property and since the condition of being a relative effective Cartier divisor is compatible with base change, the claim follows. Then it is clear to see from the definition of $AJ_{X/S} \times_S S' = AJ_{X'/S'}$.

Proof. (Proof of theorem) By the remark, we may suppose that $T = S$. Let $\varphi_L : S \to \mathrm{Pic}_{X/S}$ be an S-point corresponding to the class of a line bundle L on X[4] Then the pullback $AJ^{-1}(\varphi_L)$ by definition is the functor which we will denote $D_{[L]}$ that takes an S-scheme T to the set of relative effective Cartier divisors $D \subset X_T$ such that $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^*M$ where M is some line bundle on T.

Since f is a universal algebraic fiber space, $f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X_T)$ is injective thus if M' is some other line bundle such that $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^*M'$, then $f_T^*M \cong f_T^*M'$ so $M \cong M'$. Thus M is unique up to isomorphism and D corresponds to a regular section s of $L_T \otimes f_T^*M$. Equivalently, s is a regular section of $(f_T)_*(L_T \otimes f_T^*M)$. By the proposition, since L is flat there exists a coherent sheaf $\mathcal{Q}$ along with a functorial isomorphism

$$f_*(L \otimes f^*\mathcal{G}) = \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

for all quasi-coherent $\mathcal{G}$ on S. We will show next time that $\mathcal{E} = \mathcal{Q}$ is our required sheaf. □

[4]Here is where we are using the assumption that σ has a section so that T-points correspond to actual line bundles on X_T. Otherwise, we would need to sheafify and thus T-points would correspond to line bundles on some cover $T' \to T$.