Lecture 15 - 16: The Abel-Jacobi map (cont.), boundedness, quotients by equivalence relations

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1 $AJ_{X/S}$ is representable by projective bundles

Recall last time we defined the Abel-Jacobi map

$$AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$$

by $(D \subset X_T) \mapsto \mathcal{O}_{X_T}(D)$. We are proving the following.

Theorem 1. *Suppose $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Suppose further that the fibers of f are geometrically integral. Then the Abel-Jacobi map $AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$ is representable by a projective bundle. More precisely, for any scheme T and T-point $\varphi_L : T \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X_T, there exists a coherent sheaf $\mathcal{E}$ on T such that the pullback $AJ^{-1}_{X/S}(T) \to T$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$ over T. Moreover, if $R^1(f_T)_*L = 0$, then (1) $(f_T)_*L$ commutes with base change, (2) $\mathcal{E}$ and $(f_T)_*L$ are locally free and dual to each other, and (3) the formation of $\mathcal{E}$ commutes with base change. In particular, if $R^1(f_T)_*L = 0$ for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms.*

Proof. We have reduced to the case that $T = S$ and are considering an S-point $\varphi_L : S \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X. Let $D_{[L]}$ denote the fiber product $AJ^{-1}_{X/S}(\varphi_L)$. We saw that T-point of $D_{[L]}$ corresponds to a line bundle M on T as well as a regular section of $L_T \otimes f_T^* M$. Sections of this sheaf are the same as sections of $(f_T)_*(L_T \otimes f_T^* M)$ so we are led to consider the universal coherent sheaf $\mathcal{Q}$ on S such that

$$f_*(K \otimes f^*\mathcal{G}) = \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

for all quasi-coherent $\mathcal{G}$ on S.

We want to take $\mathcal{G}$ to be g_*M for $g : T \to S$ the structure morphism[1]. Since L and M are locally free, we have the projection formula:

$$g_*(L_T \otimes_{\mathcal{O}_{X_T}} f_T^* M) = g_*(g^*L \otimes_{\mathcal{X}_T} f_T^* M) = L \otimes_{\mathcal{O}_X} g_* f_T^* M.$$

Now f is flat so by flat base change, we have $g_* f_T^* M \cong f^* g_* M$. Putting this together, we get

$$\begin{aligned} H^0(L_T \otimes_{\mathcal{O}_{X_T}} f_T^* M) &= H^0(L \otimes_{\mathcal{O}_X} f^* g_* M) = H^0(f_*(L \otimes_{\mathcal{O}_X} f^* g_* M)) \\ &= H^0(\mathcal{H}om_S(\mathcal{Q}, g_* M)) = \mathrm{Hom}_S(\mathcal{Q}, g_* M) = \mathrm{Hom}_T(\mathcal{Q}_T, M). \end{aligned}$$

[1]Here we have to assume that g is qcqs so that this pushforward is quasi-coherent. You can convince yourself that it is enough to prove representability in the category of qcqs S-schemes.

In fact if one is more careful about the construction of $\mathcal{Q}$, one can show that it commutes with arbitrary base change so that $(f_T)_*(L_T \otimes f_T^* M) = \mathcal{H}om_T(\mathcal{Q}_T, M)$ as sheaves rather than just global sections, that is, the universal sheaf from the proposition for L_T over T is the pullback of the one for L over S.

Now the condition that a section s of $L_T \otimes_{\mathcal{O}_{X_T}} f_T^* M$ is a regular section is equivalent to $s_t \in H^0(X_t, L_t)$ being nonzero for each $t \in T$ and thus the corresponding morphism $u_s : \mathcal{Q}_T \to M$ must be nonzero at each fiber over $t \in T$. Since M is a line bundle, $M \otimes k(t)$ is a rank 1 vector space and so $u_s \otimes k(t)$ is nonzero if and only if it is surjective. By Nakayama's lemma, this implies u_s is surjective as a map of sheaves for all $t \in T$[2]. Thus $u_s : \mathcal{Q}_T \to M$ is a rank 1 locally free quotient of $\mathcal{Q}_T$. By definition, this is a T point of the projective bundle $\mathbb{P}(\mathcal{Q})$ over S.

On the other hand, given a T-point of $\mathbb{P}(\mathcal{Q})$, we can reverse the equalities above to obtain a locally free quotient $u : \mathcal{Q}_T \to M$ corresponding to a section $s : \mathcal{O}_{X_T} \to L_T \otimes f_T^* M$ which is nonzero on every fiber and thus regular. Therefore the vanishing subscheme $D \subset X_T$ of s satisfies that for all $t \in T$, $D_t \subset X_t$ is a Cartier divisor. We need to check that $D \to T$ is flat so that it is a relative effective Cartier divisor. Then by construction $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^* M$ and so $D \subset X_T$ gives a T-point of $D_{[L]}$. For flatness, we have the following lemma.

Lemma 1. *Let $f : X \to S$ be a flat morphism of finite type over a Noetherian scheme S and let $D \subset X$ be a closed subscheme such that for each $s \in S$, $D_s \subset X_s$ is an effective Cartier divisor. Then $D \to S$ is flat.*

Proof. Let $x \in D \subset X$ with $s = f(x)$. We need to show that $\mathcal{O}_{D,x}$ is a flat $\mathcal{O}_{S,s}$-module. By the local criterion for flatness, this is equivalent to the vanishing of $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{D,x})$. Consider the long exact sequence associated to the ideal sequence

$$0 \to \mathcal{I}_{D,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_{D,x} \to 0.$$

We have

$$\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{X,x}) \to \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{D,x}) \to I_{D,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s) = \mathcal{O}_{X_s,x}.$$

Since the first term is zero by flatness of $X \to S$, the required vanishing would follow from injectivity of the last map. To see this injectivity, let $\bar{f}_x \in \mathcal{O}_{X_s,x}$ be a regular element cutting out D_s at $x \in X_s$ and let $f_x \in \mathcal{O}_{X,x}$ be a lift. Now multiplication by f_x induces a map $\mathcal{O}_{X,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s)$ which is injective with image $I_{D_s,x}$. Thus we have an injective map which factors as

$$\mathcal{O}_{X,x} \otimes k(s) \to \mathcal{I}_{D,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s)$$

where the first map is a surjection and so the required map is an injection. □

This shows that $\mathcal{E} = \mathcal{Q}$ is our required sheaf so that $AJ_{X/S}^{-1}(\varphi_{[L]}) \to S$ is representable by the projective bundle $\mathbb{P}(\mathcal{E})$. If $R^1 f_* L = 0$, then by the previous corollary, $\mathcal{Q}$ and $f_* L$ are locally free, dual to eachother, and commute with basechange. In particular, if this holds for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms since a projective bundle is smooth over the base when $\mathcal{E}$ is locally free. □

[2]Here we have to use a finite presentation trick to reduce to T Noetherian as usual.

2 Boundedness

Definition 1. *We say the a moduli functor $F : Sch_S \to Set$ is bounded, or that the objects parametrized by F form a bounded family, if there exists a finite type scheme T over S as well as a T-point $\xi \in F(T)$ such that for any field $t : \mathrm{Spec}\, k \to S$ and k-point $\xi_t \in F(k)$, there exists a field extension k'/k and a k'-point $t' \in T(k')$ such that $\xi|_{t'} = \xi_t \otimes_k k'$.*

Intuitively, a bounded moduli problem, or a bounded family of geometric objects, is one where there exists a family $f : U \to T$ over a finite type base scheme such that every isomorphism class of objects in our moduli problem, appears as a fiber of f. In particular, if F is representable by some fine moduli space $\mathcal{M}$, then this induces a *surjective* morphism $T \to \mathcal{M}$ which exhibits the fine moduli space as being finite type over $\mathcal{M}$. Essentially, boundedness is a way of showing our moduli spaces are finite type.

Example 1. *Let $F = H^P_{X/S}$ be the Hilbert functor for P a fixed Hilbert polynomial and $f : X \to S$ a projective morphism over a Noetherian scheme. Then the boundedness of F was a result of uniform CM regularity which allowed us to embed F into a fixed Grassmannian, which is of finite type.*

Now let us consider our situation for the Picard functor: $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Then for any $T \to S$ and any line bundle L on X_T, L is flat over T. Therefore the Hilbert polynomial $P_{L_t}(d)$ is constant for $t \in T$ so the relative Picard functor can be written as a union

$$\bigsqcup_P \mathrm{Pic}^P_{X/S}.$$

Our goal is for each of these components $\mathrm{Pic}^P_{X/S}$ to be bounded. As with the case of the Hilbert functor, this boils down to a uniform CM regularity result.

Theorem 2. *(SGA 6, Exp XIII) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S. Suppose the fibers of f are geometrically integral and of equal dimension r and fix a Hilbert polynomial P. Then there exists an integer m such that for any field k and k-point $\xi \in \mathrm{Pic}^P_{X/S}(k)$ corresponding to a line bundle L on X_k, L is m-regular.*

Proposition 1. *For each Hilbert polynomial P, there exists an m such that the Abel-Jacobi map $AJ^{P(d+m)}_{X/S} : CDiv^{P(d+m)}_{X/S} \to \mathrm{Pic}^{P(d+m)}_{X/S}$ is the projectivization of a locally free sheaf. In particular it is a smooth, proper surjection.*

Proof. Pick m so that L on X_k is m-regular for each k-point of $\mathrm{Pic}^P_{X/S}$ and consider the Abel-Jacobi map for $P(d+m)$. For any T-point of $\mathrm{Pic}^{P(d+m)}_{X/S}$ corresponding to L on X_T, $L(-m)$ has Hilbert polynomial P, and in particular is m-regular. Therefore

$$H^i(X_t, L|_{X_t}) = 0 \;\; i \geq 1$$

for all $t \in T$. By cohomology and base change, $R^i(f_T)_*L = 0$ for all $i \geq 1$ and so $(f_T)_*L$ is locally free of rank $P(m)$ and $AJ^{P(d+m)}_{X/S} \times_{\mathrm{Pic}^{P(d+m)}_{X/S}} T \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is the locally free sheaf $((f_T)_*L)^\vee$ on T. □

Fact 1. *For each $m \in \mathbb{Z}$, twisting by $\mathcal{O}_X(m)$ induces an isomorphism*

$$\mathrm{Pic}^{P(d)}_{X/S} \cong \mathrm{Pic}^{P(d+m)}_{X/S}.$$

Corollary 1. *The functor* $\mathrm{Pic}^P_{X/S}$ *is bounded.*

Proof. By the Proposition, for each P, there exists an m such that the Abel-Jacobi map for $P(d+m)$ is surjective. Since $CDiv^{P(d+m)}_{X/S}$ is of finite type, so $\mathrm{Pic}^{P(d+m)}_{X/S}$ is bounded. By the previous fact, this functor is isomorphic to $\mathrm{Pic}^{P(d)}_{X/S}$ and so we are done. □

Now let us say a few words about the proof of the uniform CM-regularity theorem for the Picard functor. The proof is very similar to the prevoius uniform CM-regularity theorem for the Quot funcotr. Recall that the idea there was to induct on the dimension of the ambient projective space and restrict to a hyperplane section. More precisely, the proof there showed the following.

Proposition 2. *Let $\mathcal{F}$ be a coherent sheaf on a projective variety X over a field* $\mathrm{Spec}\, k$*. Suppose we have an exact sequence*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$$

given by restricting to a hyperplane section H such that $\mathcal{F}_H$ is m-regular. Then:

(a) $H^i(X, \mathcal{F}(n)) = 0$ *for $n \geq m - i$, and*

(b) the sequence $\{\dim H^1(X, \mathcal{F}(n)\}$ *is monotonically decreasing to zero for $n \geq m-1$.*

In particular, $H^1(X, \mathcal{F}(n)) = 0$ *for* $n \geq (m-1) + \dim H^1(X, \mathcal{F}(m-1))$ *so that $\mathcal{F}$ is* $[m + \dim H^1(X, \mathcal{F}(m-1))]$*-regular.*

In the previous incarnation of uniform CM-regularity the only place where we used that $\mathcal{F}$ was a subsheaf of $\mathcal{O}_X^{\oplus r}$ was to bound $H^0(X, \mathcal{F}(m))$ in terms of the Hilbert polynomial P. By the above proposition, we get that

$$\dim H^1(X, \mathcal{F}(m-1)) = H^0(X, \mathcal{F}(m-1)) - P(m-1)$$

and so a uniform bound for $H^0(X, \mathcal{F}(m-1))$ gives us a uniform bound for the regularity of $\mathcal{F}$. Thus, the main technical part of the proof of the theorem is to bound the dimension of the space of global sections of a line bundle with fixed Hilbert polynomial.

Definition 2. *The degree of a projective variety $X \subset \mathbb{P}^n_k$ of pure dimension r is given by the intersection number H^r where H is a section of $\mathcal{O}_X(1)$.*

The idea then is to relate the Hilbert polynomial, the degree, and the space of global sections and their restrictions to hyperplane sections (for the inductive step!) using Grothendieck-Riemann-Roch and Serre duality. We won't say more about the general proof here but let us consider the easier case of X a smooth projective curve.

2.1 Boundedness of Picard for smooth projective curves

For C an integral projective curve over a field k, we can define the arithmetic genus

$$p_a := \dim H^1(C, \mathcal{O}_C).$$

If $C^\nu \to C$ is the normalization of C, we define the geometric genus by

$$p_g(C) := p_a(C^\nu).$$

When C is already normal, and thus regular, we have $p_g = p_a$ and we simply call this the genus $g = g(C)$. On such a C we have the canonical bundle

$$\omega_C := \Omega^1_{C/k}.$$

Recall the statement of Serre duality.

Theorem 3. *(Serre duality) Let C be a projective, integral and regular curve over a field. Then for any locally free coherent sheaf $\mathcal{E}$ on C, there is a natural isomorphism*

$$H^1(C, \mathcal{E})^\vee \cong H^0(C, \omega_C \otimes \mathcal{E}^\vee).$$

We also have the Riemann-Roch theorem which allows us to compute the Hilbert polynomial of a line bundle.

Theorem 4. *(Riemann-Roch) Let L be a line bundle on C a projective, integral and regular curve over a field. Then*

$$\chi(L) := \dim H^0(C, L) - \dim H^1(C, L) = \deg(L) - g + 1.$$

Note that for C an integral, regular, projective curve, the degree of C in the sense of Definition 2 above is the same as the degree of $\mathcal{O}_C(1)$. Then by Riemann-Roch, for L any line bundle, we have

$$P_L(m) = Dm + \deg(L) - g + 1$$

where $D = \deg(_{\mathcal{C}}(\infty)) = \deg(\mathcal{C})$. Therefore, the Hilbert polynomial of a line bundle depends only on the degree $\deg(L)$.

On the other hand, a Cartier divisor $D \subset C$ is simply a zero dimensional subscheme and it has constant Hilbert polynomial $d = \dim \mathcal{O}_D$ and the degree of $\mathcal{O}_C(D)$ is simply d. Thus, we can label the components of the Picard functor by the degree $d = \deg L$ and the Abel-Jacobi map takes the form

$$AJ^d_{X/S} : CDiv^d_{C/k} = \mathrm{Hilb}^d_{C/k} \to \mathrm{Pic}^d_{C/k}.$$

Applying Riemann-Roch and Serre duality to $L = \omega_C$, we get that $\dim H^0(C, \omega_C) = g$ and $\deg(\omega_C) = 2g - 2$. Then if L is a line bundle with $\deg L > 2g - 2$,

$$\dim H^1(C, L) = \dim H^0(C, \omega_C \otimes L^{-1}) = 0$$

since $\deg(\omega_C \otimes L^{-1}) < 0$. Therefore the Abel-Jacobi map is a smooth projective bundle for $d > 2g - 2$, in fact equal to the projectivization $\mathbb{P}((\pi_*\mathcal{L})^\vee)$ where $\mathcal{L}$ is the universal line bundle on $C \times \mathrm{Pic}^d_{C/k}$ and π is the second projection. In particular, this gives boundedness.

3 Quotients by flat and proper equivalence relations

The last technical ingredient we need before we can prove representability of the Picard functor is the existence of quotients by finite equivalence relations for quasi-projective schemes. We begin with some generalities on categorical quotients.

Let $\mathcal{C}$ be a category with fiber products and a terminal object $*$. An equivalence relation on an object X of $\mathcal{C}$ is an object R along with a morphism $R \to X \times_* X$ such for each object T, the map of sets $R(T) \subset X(T) \times X(T)$ is the inclusion of an equivalence relation on the set $X(T)$. The two projections give us two morphisms $p_i : R \to X$ from an equivalence relation to X.

Definition 3. *A categorical quotient of X by the equivalence relation R is an object Z as well as a morphism $u : X \to Z$ such that $u \circ p_1 = u \circ p_2$ such that (Z, u) is initial with respect to this property. That is, for any $f : X \to Y$ such that $f \circ p_1 = f \circ p_2$, there exists a unique morphism $g : Z \to Y$ such that f factors through u, $f = g \circ u$.*

If such a pair (Z, u) exists, then it is unique up to unique isomorphism and is denoted X/R. If X/R exists, we say that it is an effective quotient if the natural map

$$R \to X \times_{X/R} X$$

is an isomorphism.

More generally, we can consider maps $R \to X \times_* X$ that are not necessarily monomorphisms but such that the image of $R(T) \to X(T) \times X(T)$ is an equivalence relation. In this can we can replace R with its image in $X \times_* X$ if it exists to reduce to the previous situation.[3]

Example 2. *(The case of a group quotient) Let G be a S-group scheme acting on an S-scheme X. Then the action is given by a morphism $m : G \times_S X \to X$ and the product $m \times pr_X : G \times_S X \to X \times_S X$ is an equivalence relation on X in the category of S-schemes. If an effective quotient exists, we will denote it X/G. Note that in this case, the fibers of the natural map $u : X \to X/G$ are exactly the orbits of G.*

Example 3. *(A non-effective quotient) Consider $\mathbb{A}^1$ over an algebraically closed field $k = \bar{k}$. The group $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling and there are two orbits, $U = \mathbb{A}^1 \setminus \{0\}$ and $\{0\}$. Now $\mathbb{A}^1 \to$* Spec *k is a categorical quotient but the fibers of this map are not orbits so the quotient isn't effective.*

Given an equivalence relation $R \to X \times_S X$ on an S-scheme X, we say that R has property $\mathcal{P}$ for any property of morphisms if the morphisms $p_i : R \to X$ have this property.

Theorem 5. *Let $X \to S$ be a quasi-projective scheme over a Noetherian scheme S and suppose that $R \to X \times_S X$ is a flat and proper equivalence relation. Then an effective quotient X/R exists and moreover it is a quasi-projective S-scheme.*

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[3]This doesn't make a difference for us now but when we consider the more general category of algebraic stacks, taking quotients by a finite map $R \to X \times_S X$ versus its image $R' \subset X \times_S X$ is exactly the difference between a stack quotient and its coarse moduli space.