Lecture 17-18: Sheaves, quotients, representability of the Picard functor

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1 fpqc Descent

Given a morphism $p : S' \to S$, we can consider the pullback functor p^*.

$$\begin{aligned} p^* : QCoh(S) &\to QCoh(S') \\ \mathcal{F} &\mapsto p^*\mathcal{F} \end{aligned}$$

Denoting by $q_i : S'' := S' \times_S S' \to S'$ the two projections, then the sheaf $p^*\mathcal{F}$ carries a natural isomorphism

$$\varphi : q_1^* p^* \mathcal{F} \to q_2^* p^* \mathcal{F}.$$

given by the isomorphism of functors

$$q_1^* \circ p^* \cong (p \circ q_1)^* = (p \circ q_2) \cong q_2^* \circ p^*.$$

Now we can consider the various projections from the triple product,

$$q_{ij} : S''' := S' \times_S S' \times_S S' \to S''.$$

Then for any sheaf $\mathcal{F}$ on S, we have the following commutative diagram.

$$\begin{array}{ccccc} q_{12}^* q_1^* p^* \mathcal{F} & \xrightarrow{q_{12}^*\varphi} & q_{12}^* q_2^* p^* \mathcal{F} = q_{23}^* q_1^* p^* \mathcal{F} & \xrightarrow{q_{23}^*\varphi} & q_{23}^* q_2^* p^* \mathcal{F} \\ \| & & & & \| \\ q_{13}^* q_1^* p^* \mathcal{F} & & \xrightarrow{\qquad q_{13}^*\varphi \qquad} & & q_{13}^* q_2^* p^* \mathcal{F} \end{array}$$

Said succinctly, we have the cocycle condition

$$q_{13}^*\varphi = q_{23}^*\varphi \circ q_{12}^*\varphi. \tag{1}$$

Let $QCoh(p : S' \to S)$ denote the category of pairs $(\mathcal{F}', \varphi)$ where $\mathcal{F}'$ is a quasi-coherent sheaf on S' and $\varphi : q_1^*\mathcal{F}' \to q_2\mathcal{F}'$ is an isomorphism satisfying the cocycle condition (1). Then p^* gives a functor

$$p^* : QCoh(S) \to QCoh(p : S' \to S).$$

The question of descent is the question of when p^* is an equivalence of categories. We say that quasi-coherent sheaves satisfy descent along p or that descent holds for p when this functor is an equivalence.[1]

[1] The cateogry $QCoh(p : S' \to S)$ is sometimes called the category of descent data and the descent data in the image of p^* is called effective.

Example 1. *Suppose $\{U_i\}_{i\in I}$ is a Zariski open cover of S and let $p : S' = \coprod_{i\in I} U_i \to S$. Then S'' is the disjoint union of intersections $U_i \cap U_j$, S''' is the disjoint union of triple intersections, and the cocycle condition is the usual cocycle condition for gluing sheaves so quasi-coherent sheaves satisfy descent along p.*

Definition 1. *A morphism $p : S' \to S$ is fpqc[2] if it is faithfully flat and each point $s' \in S'$ has a quasi-compact open neighborhood $U \subset S'$ with $f(U)$ an open affine subset of S. A morphism $p : S' \to S$ if fppf if it is faithfully flat and of finite presentation.*

fpqc and fppf morphisms satisfy many nice properties.

Fact 1. *(i) the property of being fpqc or fppf is compatible under base-change and composition;*

(ii) if $p : S' \to S$ is fpqc, then S has the quotient topology of S' by p. That is, $U \subset S$ is open if and only if $f^{-1}(U) \subset S'$ is open;

(iii) an open faithfully flat morphism is fpqc;

(iv) an fppf morphism is open, and in particular, fpqc.

Many properties of schemes (resp. morphisms) are fpqc local (resp. fpqc local on the target), meaning they can be checked after pulling back by an fpqc morphism. This includes the propeties we defined for representability of morphisms in a previous lecture. The following is Grothendieck's main theorem of descent.

Theorem 1. *Let $p : S' \to S$ be an fpqc morphism. Then quasi-coherent sheaves satisfy descent by p:*

$$p^* : QCoh(S) \cong QCoh(p : S' \to S).$$

We can also talk about the question of descent for other objects on S. For example we can consider the case of schemes $X \to S$. Given such an X we can pull it back along p to obtain $X' = p^*X \to S'$ an S'-scheme with an isomorphism $\varphi : q_1^*X' \to q_2^*X'$ of S''-schemes satisfying the cocycle condition on the triple fiber product S'''. We have a category of S'-schemes with descent data $Sch_{S'\to S}$ consisting of (X', φ) where $\varphi : q_1^*X \to q_2^*X$ is an isomorphism. Then p^* gives a functor

$$p^* : Sch_S \to Sch_{S'\to S}$$

and we can ask when p^* is an equivalence.

Corollary 1. *Let $p : S' \to S$ be an fpqc morphism. Let Aff_S be the category of affine S-schemes and $Aff_{S'\to S}$ the category of affine S'-schemes with descent data. Then*

$$p^* : Aff_S \to Aff_{S'\to S}$$

is an equivalence. That is, affine S-schemes satisfy fpqc descent. In particular, closed subschemes of S satisfy fpqc descent.

[2]"faithfully flat and quasi-compact"

Proof. The relative spec functor Spec_S gives an equivalence of categories between affine S-schemes and quasi-coherent $\mathcal{O}_S$-algebras over S. Moreover, $p^* : QCoh(S) \to QCoh(S')$ is compatible with tensor products. Thus, it sends quasi-coherent $\mathcal{O}_S$-algebras to $\mathcal{O}_{S'}$-algebras and the canonical isomorphism $\varphi : q_1^* p^* \mathcal{A} \to q_2^* p^* \mathcal{A}$ is an algebra homomorphism. Thus, the equivalence

$$p^* : QCoh(S) \to QCoh(p : S' \to S)$$

restricts to an equivalence on the subcategories of algebra objects which by the Spec_S equivalence gives us the first claim. For the second statement, closed subschemes of S correspond to affine morphisms $f : X \to S$ such that $\mathcal{O}_S \to f_* \mathcal{O}_X$ is a surjection, or equivalently, algebras such that the canonical map $\mathcal{O}_S \to \mathcal{A}$ is a surjection. As before, since p^* is an equivalence onto the category of quasi-coherent sheaves with descent data, $\mathcal{O}_S \to \mathcal{A}$ is a surjection if and only if $\mathcal{O}_{S'} \to \mathcal{A}'$ is a surjection so closed subschemes descend to closed subschemes. □

More generally, for any fpqc morphism, it is a fact that

$$p^* : Sch_S \to Sch_{S' \to S}$$

is fully faithful. Essential surjectivity (ie effectivity of descent data) is more subtle but the affine case above suggests that one should restrict to desent data (X, φ) for which there exists an open affine cover of X by U such that φ restricts to an isomorphism $q_1^* U \to q_2^* U$ for each U.

2 Grothendieck topologies and sheaves

Given a category $\mathcal{C}$ which has pullbacks, a collection of morphisms $\mathcal{T}$ generates[3] a *Grothendieck topology* if

(1) any isomorphism $X \to Y$ is contained in $\mathcal{T}$;

(2) for any $U \to X$ in $\mathcal{T}$ and any $X' \to X$, the pullback $U' \to X'$ is in $\mathcal{T}$;

(3) For any $U \to X$ and $V \to U$ in $\mathcal{T}$, the composition $V \to X$ is in $\mathcal{T}$.

Example 2. *If Top is the category of topological spaces, then the collection of morphisms of the form $U = \bigsqcup_{i \in I} U_i \to X$ where $\{U_i\}$ is an open cover X generate a grothendieck topology. Similarly, replacing Top by Sch_S and open cover by Zariski open cover, we obtain the Zariski topology.* [4]

Definition 2. *Let $\mathcal{C}$ be a category with Grothendieck topology generated by $\mathcal{T}$. A presheaf $F : \mathcal{C} \to Set$ is a sheaf for $\mathcal{T}$ if*

1. *for any collection of objects $\{T_i\}$, $F(\bigsqcup T_i) = \prod F(T_i)$, and*

[3] Technically, the collection $\mathcal{T}$ is not the Grothendieck topology, but rather a Grothendieck pre-topology, and different pre-topologies may generate the same topology. The notion of sheaves which we will define shortly depends only on the topology not the pre-topology but we won't need this distinction here. One should think of $\mathcal{T}$ is a sub-base for the topology.

[4] Note that here we are using a convenient notational trick of replacing an open covering $\{U_i\}$ with their disjoint union $\bigsqcup U_i$ mapping to X.

2. *for any object X and morphism $U \to X$ in $\mathcal{T}$, the sequence*

$$F(X) \to F(U) \rightrightarrows F(U \times_X U)$$

is an equalizer where the maps are induced by the two projections.

The topologies we will consider now, beyond the Zariski topology, are the fpqc and fppf topologies, where $\mathcal{T}$ is the collection of fpqc, respectively fppf morphisms. The main theorem is the following.

Theorem 2. *Let F be a representable functor $Sch_S \to Set$. Then F is a sheaf for the fpqc topology.*

We will leave this as an exercise, with the hint that this follows from fpqc descent. More precisely, one uses the fact that for an fpqc morphism $p : S' \to S$, the functor $p^* : Sch_S \to Sch_{S' \to S}$ is fully faithful.

Finally, we recall the notion of sheafification. Let F be any presheaf on a category $\mathcal{C}$ with Grothendieck topology generated by a collection $\mathcal{T}$. For any $p : U \to X$ in $\mathcal{T}$, we define

$$H^0(F, p) = Eq(F(U) \rightrightarrows F(U \times_X U)).$$

Now we define the presheaf F^+ by

$$F^+(X) = \operatorname{colim}_{(p:U \to X) \in \mathcal{T}} H^0(F, p).$$

There is a natural morphism of presheaves $F \to F^+$. We have the following theorem.

Theorem 3. *The construction $F \to (F \to F^+)$ is functorial in F. Moreover, for any F, the presheaf F^{++} is a sheaf. Moreover, it is universal for sheaves receiving a map from F.*

We call F^{++} the sheafification of F for the topology generated by $\mathcal{T}$.

3 Quotients by flat and proper equivalence relations

We now return to the existence of quotients by flat and proper equivalence relations. Given an equivalence relation $R \to X \times_S X$ on an S-scheme X, one can consider the categorical quotient in the category of fppf sheaves on Sch_S. In this category, all quotients exist. Indeed, we can define the quotient $(X/R)_{fppf}$ as the fppf-sheafification of the presheaf

$$T \mapsto X(T)/R(T)$$

where the latter denotes the quotient of the set $X(T)$ by the set theoretic equivalence relation $R(T)$.

Lemma 1. *Let $f : X \to Z$ be an fppf morphism of S-schemes and let $R = X \times_Z X \subset X \times_S X$. Then Z is an effective quotient of X by R in the category of schemes, and moreover, Z represents the fppf-sheafification $(X/R)_{fppf}$.*

Proof. Let $g : X \to Y$ be any morphism such that $g \circ p_1 = g \circ p_2$ where $p_i : R \to X$ are the two projections. Then as a morphism of sheaves for the fppf topology, g factors uniquely through $(X/R)_{fppf}$ as $X \to (X/R)_{fppf} \to Y$. We conclude that if $(X/R)_{fppf}$ is representable by a scheme, then it must be the categorical quotient of X by R. On the other hand, by fppf descent, in particular, fully faithfulness of the functor

$$f^* : Sch_Z \to Sch_{X \to Z},$$

Z represents the functor $(X/R)_{fppf}$ and so Z is a categorical quotient of X by R. By assumption, $R = X \times_Z X$ so the quotient is effective. □

Remark 1. *This same analysis could have been carried out with the fppf topology replaced by the fpqc topology.*

Theorem 4. *Let $f : X \to S$ be a quasi-projective scheme over a Noetherian scheme S and let $R \subset X \times_S X$ be a flat and proper equivalence relation on X. Then an effective quotient X/R exists and it is a quasi-projective S-scheme. Moreover, the map $q : X \to X/R$ is fppf.*

Proof. Since $R \to X$ is flat and proper over a Notherian scheme, and so in particular, of finite presentation, there are a finite number of Hilbert polynomials $\{P_1, \dots, P_n\}$ such that the fibers of $R \to X$ have Hilbert polynomial $P = P_i$ for some i. Let $H = \bigsqcup \mathrm{Hilb}^{P_i}_{X/S}$ be the quasi-projective S-scheme obtained as the union of these components of the Hilbert scheme of X/S and let $\mathcal{Z} \subset X \times_S H$ be the universal family of subschemes over H. Then $R \subset X \times_S X$ gives an X-point of H, that is, a morphism

$$g : X \to H$$

such that $g^*\mathcal{Z} = R$.

Let $\Gamma_g \subset X \times_S H$ be the graph of g. Since $H \to S$ is separated, $\Gamma_g \subset X \times_S H$ is a closed embedding. Now for any T, let $x_1, x_2 \in X(T)$ be two T-points which by the isomorphism $\Gamma_g \to X$ can be identified with T-points of the graph. Now we have

$$(x_1, x_2) \in R(T) \iff (x_1, gx_2) \in \mathcal{Z}(T) \iff gx_1 = gx_2.$$

The first equality follows by the fact that $g^*\mathcal{Z} = R$ and the second from the fact that g is the second projection from the graph and the properties of an equivalence relation. In particular, since $(x_1, x_1) \in R(T)$, then $(x_1, gx_1) \in \mathcal{Z}(T)$ so $\Gamma_g \subset \mathcal{Z}$ is a closed subscheme of $\mathcal{Z}$.

Now $\mathcal{Z} \to H$ is an fppf morphism. We claim that as subschemes of the fiber product $\mathcal{Z} \times_H \mathcal{Z}$, $\Gamma_g \times_H \mathcal{Z} = \mathcal{Z} \times_H \Gamma_g$. A T-point of either, by the string of equalities above and the definition of a graph, corresponds to a pair $(x_1, x_2) \in R(T)$ and so we conclude the required equality. Then by fpqc descent of closed subschemes, $\Gamma_g \subset \mathcal{Z}$ descends to a closed subscheme $Y \subset H$ with an fppf morphism $\Gamma_g \to Y$. By definition of the graph this can be identified with the morphism $g : X \to H$ and so g factors through an fppf morphism $g : X \to Y$. By the lemma, this fppf morphism is an effective categorical quotient of X by $X \times_Y X$ but again by the above string of equalities, this fiber product is just R. Therefore $g : X \to Y$ is an effective categorical quotient of X by R. Finally, Y is a closed subscheme of the quasi-projective S-scheme H so Y is quasi-projective.

□

4 Representability of the Picard functor

We are now ready to prove the main representability result, assuming the above theorem on quotients by flat and proper equivalence relations. We need the following preliminary result.

Proposition 1. *Let $f : X \to S$ be a flat projective universal algebraic fiber space over a Noetherian scheme S. Suppose S has a section $\sigma : S \to X$ and the fibers of f are geometrically integral. Then $\mathrm{Pic}_{X/S}$ is an fppf sheaf.*

Proof. Under these assumptions[5], the functor $\mathrm{Pic}_{X/S}$ is isomorphic to the functor of σ-rigidified line bundles $\mathrm{Pic}_{X/S,\sigma}$. Now using fppf descent one can check that this latter functor is an fppf sheaf. □

Theorem 5. *Let $f : X \to S$ be a flat projective universal algebraic fiber space over a Noetherian scheme S. Suppose f has a section $\sigma : S \to X$ and that the fibers of f are geometrically integral. Then for each Hilbert polynomial P, the functor $\mathrm{Pic}^P_{X/S}$ is representable by a quasi-projective S-scheme.*

Proof. By uniform CM regularity, there exists an m such that for any k-point of $\mathrm{Pic}^P_{X/S}$ corresponding to a line bundle L on X_k, then $H^i(X_k, L(m)) = 0$ for all $i > 0$. In particular, the Abel-Jacobi map for $P_1(d) := P(d+m)$ is representable by smooth and proper surjections. Let P_2 be the Hilbert polynomial of the component of $CDiv_{X/S}$ such that for any $D \subset X_T$ with Hilbert polynomial P_2, $\mathcal{O}_{X_T}(D)$ has Hilbert polynomial P_1 and let us denote

$$\mathcal{D}(P_2) := CDiv^{P_2}_{X/S}.$$

Let R denote the fiber product $\mathcal{D}(P_2) \times_{CDiv^{P_1}_{X/S}} \mathcal{D}(P_2)$.

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{D}(P_2) \\ \downarrow & & \downarrow \\ \mathcal{D}(P_2) & \longrightarrow & \mathrm{Pic}^{P_1}_{X/S} \end{array} .$$

Then $R \to \mathcal{D}(P_2) \times_S \mathcal{D}(P_2)$ is a flat and proper equivalence relation. By the fppf sheaf condition,

$$\mathrm{Pic}^{P_1}_{X/S}$$

is the quotient in the category of fppf sheaves of $\mathcal{D}(P_2)$ by R and by the previous theorem, an effective quotient $\mathcal{D}(P_2)/R$ exists in the category of quasi-projective S-schemes so this quotient represents $\mathrm{Pic}^{P_1}_{X/S}$. Then tensoring by $\mathcal{O}_X(m)$ induces an isomorphism

$$\mathrm{Pic}^{P_1}_{X/S} \cong \mathrm{Pic}^{P}_{X/S}$$

so we conclude representability of $\mathrm{Pic}^P_{X/S}$ by a quasi-projective S-scheme. □

[5]Check which assumptions we actually need.