Lecture 19-21: Deformation theory of line bundles, compactified Jacobians of integral curves

11/11/2019, 11/13/2019 & 11/15/2019

1 Deformation theory of line bundles

Our goal now is to compute the local structure of the Picard scheme. In particular, we can ask is it regular or smooth? The first step is to compute the tangent space. Recall the following basic proposition from scheme theory.

Proposition 1. *Let Y be a scheme and $\xi : \operatorname{Spec} k \to Y$ a k-point. The tangent space $T_\xi Y$ is the the set of maps* $\operatorname{Spec} k[\epsilon] \to Y$ *where $\epsilon^2 = 0$ such that the following diagram commutes*

$$\begin{array}{ccc} \operatorname{Spec} k[\epsilon] & \longrightarrow & Y \\ \uparrow & \nearrow_{\xi} & \\ \operatorname{Spec} k & & \end{array}$$

In the case Y is the Picard scheme $\operatorname{Pic}_{X/S}$, ξ corresponds to a line bundle L on X_k and the tangent space is the fiber over $[L]$ of the map of groups

$$\operatorname{Pic}_{X/S}(k[\epsilon]) \to \operatorname{Pic}_{X/S}(k).$$

Using the group action, we can tensor by L^{-1} so that we get a new point $\xi' : \operatorname{Spec} k \to \operatorname{Pic}_{X/S}$ corresponding to the line bundle $L \otimes L^{-1} = \mathcal{O}_{X_k}$. Since tensoring by a line bundle is an isomorphism of functors, it suffices to compute the tangent space for $\mathcal{O}_{X_k}$. This is the identity of the group $\operatorname{Pic}_{X/S}(k)$ so weve deduced that the tangent space to the Picard scheme is isomorphic to the kernel of the map above. That is, we have an exact sequence

$$0 \to T_\xi \operatorname{Pic}_{X/S} \to \operatorname{Pic}_{X/S}(k[\epsilon]) \to \operatorname{Pic}_{X/S}(k).$$

Proposition 2. *The tangent space to $\xi : \operatorname{Spec} k \to \operatorname{Pic}_{X/S}$ corresponding to the line bundle $\mathcal{O}_{X_k}$ is isomorphic to*

$$H^1(X_k, \mathcal{O}_{X_k}).$$

Proof. The scheme $X_{k[\epsilon]}$ has the same underlying topological space as X_k with structure sheaf $\mathcal{O}_{X_{k[\epsilon]}} = \mathcal{O}_X[\epsilon] := \mathcal{O}_X \otimes_k k[\epsilon]$. The map

$$\operatorname{Pic}_{X/S}(k[\epsilon]) \to \operatorname{Pic}_{X/S}(k)$$

can be identified with the map

$$H^1(X_k, \mathcal{O}_{X_k}[\epsilon]^*) \to H^1(X_k, \mathcal{O}_{X_k}^*).$$

Here we are taking cohomology of sheaves of abelian groups on the underlying topological space X_k. To compute the kernel, consider the short exact sequence of sheaves of abelian groups, written multiplicatively.

$$1 \to 1 + \epsilon \mathcal{O}_{X_k} \to \mathcal{O}_{X_k}[\epsilon]^* \to \mathcal{O}_{X_k}^* \to 1.$$

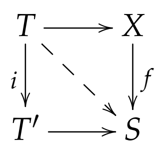
The multiplicative sheaf $1 + \epsilon \mathcal{O}_{X_k}$ is isomorphic to the sheaf of additive abelian groups $\mathcal{O}_{X_k}$ since $\epsilon^2 = 0$. Taking the long exact sequence of cohomology we get

$$H^0(X_k, \mathcal{O}_{X_k}[\epsilon]^*) \to H^0(X_k, \mathcal{O}_{X_k}^*) \to H^1(X_k, \mathcal{O}_{X_k}) \to \mathrm{Pic}_{X/S}(k[\epsilon]) \to \mathrm{Pic}_{X/S}(k).$$

The first map is surjective, so by exactness, the kernel of interest is $H^1(X_k, \mathcal{O}_{X_k})$ as claimed. □

Having computed the tangent space to $\mathrm{Pic}_{X/S}$, we can ask more generally if it is smooth over S. Recall the following definition of formally smooth.

Definition 1. *A map of schemes $X \to S$ is formally smooth if for any closed embedding of affine S-schemes $i : T \to T'$ defined by a square zero ideal, and any solid diagram as below, there exists a dotted arrow making the diagram commute.*



The advantage of formal smoothness is that it is a condition on the functor of points of $X \to S$ on Sch_S. On the other hand, we have the following lifting criterion for smoothness.

Proposition 3. *A morphism $X \to S$ is smooth if and only if it is formally smooth and locally of finite presentation.*

The conditions under which we proved representability of the Picard functor also gurantee that $\mathrm{Pic}_{X/S} \to S$ is locally of finite presentation. Thus we can check smoothness using formal smoothness. It suffices to consider the case $S = \mathrm{Spec}\, R$ is affine. Then $i : T \to T'$ corresponds to a surjection $A' \to A$ of R-algebras with kernel I satisfying $I^2 = 0$. The lifting criterion to smoothness then asks the question of when

$$\mathrm{Pic}_{X/S}(A') \to \mathrm{Pic}_{X/S}(A)$$

is surjective. Repeating the argument from the computation of the tangent space, we get the following.

Proposition 4. *Suppose $f : X \to S$ is an S-scheme which is A-flat. Then we have an exact sequence*

$$0 \to H^1(X_A, f^* I) \to \mathrm{Pic}(X_{A'}) \to \mathrm{Pic}(X_A) \to H^2(X_A, f^* I).$$

Proof. By A-flatness of X, we have that the sequence

$$0 \to f^*I \to \mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A} \to 0,$$

obtained by pulling back $0 \to I \to A' \to A \to 0$ to X, is exact. Since I is square zero, we have the following exact sequence of multiplicative groups.

$$1 \to 1 + f^*I \to \mathcal{O}^*_{X_{A'}} \to \mathcal{O}^*_{X_A} \to 1$$

Taking the long exact sequence of cohomology and using the fact that the multiplicative sheaf $1 + f^*I$ is isomorphic to the additive sheaf f^*I and that the map $H^0(X_A, \mathcal{O}^*_{X_{A'}}) \to H^0(X_A, \mathcal{O}^*_{X_A})$ is surjective concludes the proof. $\square$

Corollary 1. *Suppose $T \to T'$ is a square zero thickening of affine schemes corresponding to $A' \to A$. Then $\mathrm{Pic}(T') \to \mathrm{Pic}(T)$ is an isomorphism.*

Proof. By the proposition, we have an exact sequence

$$H^1(T, I) \to \mathrm{Pic}(T') \to \mathrm{Pic}(T) \to H^2(T, I).$$

Now I is quasi-coherent and T is affine so the first and last groups vanish. $\square$

Putting this all together, we get the following result.

Theorem 1. *Let $S = \mathrm{Spec}\ R$ be an affine Noetherian scheme and suppose that $f : X \to S$ is as in the existence theorem for the Picard scheme.*[1] *Then for any $A' \to A$ a morphism of R-algebras with square-zero kernel I, and any diagram*

$$\begin{array}{ccc} \mathrm{Spec}\ A & \xrightarrow{\xi} & \mathrm{Pic}_{X/S} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\ A' & \longrightarrow & S \end{array}$$

there exists an element $\mathrm{obs}(\xi) \in H^2(X_A, f^*I)$ such that a lift $\xi' : \mathrm{Spec}\ A' \to \mathrm{Pic}_{X/S}$ exists if and only of $\mathrm{obs}(\xi) = 0$. Moreover, in this case, the set of such lifts is a torsor[2] *for the group $H^1(X_A, f^*I)$.*

Proof. Combining the above proposition and corollary, we see that for $T \to T'$ being a square-zero thickening of affine schemes, the exact sequence of Proposition 5 becomes an exact sequence

$$0 \to H^1(X_A, f^*I) \to \mathrm{Pic}_{X/S}(A') \to \mathrm{Pic}_{X/S}(A) \to H^2(X_A, f^*I).$$

Here we have used the assumptions on $f : X \to S$ only to gurantee that $\mathrm{Pic}_{X/S}(A) = \mathrm{Pic}(X_A)/\mathrm{Pic}(A)$ and similarly for A'. Then the statement of the theorem is just a reinterpretation of exactness. Indeed ξ gives an element of $\mathrm{Pic}_{X/S}(A)$ and its image in $H^2(X_A, f^*I)$ is $\mathrm{obs}(\xi)$. Then ξ is in the image of the middle map if and only if it is in the kernel of the last map if and only of $\mathrm{obs}(\xi) = 0$. When this happens, the set of preimages of ξ under the middle map has a free and transitive action by the kernel of the middle map, which is exactly the image of $H^1(X_A, f^*I) \to \mathrm{Pic}_{X/S}(A')$. $\square$

[1] These assumptions can be relaxed for deformation theory but we keep them here for simplicity.

[2] a set with a free and transitive action of

This is an exampe of a deformation-obstruction theory, in this case for the Picard functor. The connecting map obs : $\mathrm{Pic}_{X/S}(A) \to H^2(X, f^*I)$ and the association that takes a square-zero thickening of affine schemes $A' \to A$ to the groups $H^*(X, f^*I)$ is functorial in I. The group $H^2(X, f^*I)$ is the *obstruction group* and $H^1(X, f^*I)$ is the *group of first order deformations*. The special case where $A' = k[\epsilon] \to A = k$ gives us the tangent space to $\mathrm{Pic}_{X/S}$ and, analagously, the deformation-obstruction theory can be thought of as encoding functorially the local structure of $\mathrm{Pic}_{X/S}$.

In general, one can ask whether a moduli functor admits a deformation-obstruction theory which has the features above (an obstruction group which receives an obstruction map whose image vanishes if and only if a lift exists and a deformation group under which the set of lifts is a torsor if its nonempty which are functorial in the square-zero extension $A' \to A$). This forms the basis of Artin's axiomatic approach to representability of moduli problems by algebraic spaces and stacks.

From the previous result, we have the following corollary.

Corollary 2. *Let $f : X \to S$ as in the existence theorem for the Picard scheme and suppose further that the fibers are curves. Then $\mathrm{Pic}_{X/S}$ is smooth over S of relative dimension g, the arithmetic genus of the family of curves f.*

Proof. We can suppose without loss of generality that $S = \mathrm{Spec}\ R$ is affine. Then the statement follows from the lifting criterion for smoothness (since $\mathrm{Pic}_{X/S} \to S$ is locally of finite type and S is Noetherian). To check that the lifting holds, it suffices to check that obstruction group vanishes. By the theorem this is a second coherent cohomology group which vanishes since $X \to S$ is a curve. Finally, the tangent space to a fiber over Spec $k \to S$ is computed by $H^1(X_k, \mathcal{O}_{X_k})$ which is $g = p_a$ dimensional. □

2 Jacobians of integral curves

We saw previously that when X is a smooth projective curve over a field, the Hilbert polynomials of line bundles are just indexed by the degree d and the Abel-Jacobi map is given as

$$AJ^d_{X/k} : \mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}^d_{X/k}$$

from the Hilbert scheme of zero dimensional subschemes with Hilbert polynomial constant d, that is, subschemes of length d, to the component of the Picard scheme of degree d line bundles. Moreover, we saw using Riemann-Roch and Serre duality that for $d > 2g - 2$, AJ^d_X is a smooth projective bundle with fiber dimension $d - g$. By the results of the previous section on deformation theory, we also know that $\mathrm{Pic}^d_{X/k}$ is smooth. In particular, since the genus g is constant in flat families, this holds in the relative setting so that

$$AJ^d_{X/S} : \mathrm{Hilb}^d_{X/S} \to \mathrm{Pic}^d_{X/S}$$

is a smooth projective bundle of rank $d - g$ for $d > 2g - 2$ whenever $f : X \to S$ is a smooth integral projective one dimensional universal algebraic fiber space with a section over a Noetherian base. Moreover, we saw in the smooth case $\mathrm{Pic}^d_{X/S}$ is in fact proper.

Definition 2. *The Jacobian* $\mathrm{Jac}_{X/S} = \mathrm{Pic}^0_{X/S}$ *is the degree zero component of the Picard scheme.*

In particular, when $f : X \to S$ is a smooth curve with the assumptions above $\mathrm{Jac}_{X/S} \to S$ is a smooth and proper group scheme over S so it is an abelian scheme. Moreover, in this case, if $f : X \to S$ has a section $\sigma : S \to X$, then twisting by $\mathcal{O}_X(\sigma(S))$ gives an isomorphism $\mathrm{Pic}^n_{X/S} \cong \mathrm{Pic}^{n+1}_{X/S}$ and so each component is isomorphic to the Jacobian.[3]

In the special case when $S = \mathrm{Spec}\ \mathbb{C}$ it is the abelian variety corresponding to the g-dimensional complex analytic torus

$$H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

Indeed using the exponential sheaf sequence

$$0 \longrightarrow 2\pi i\mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

one can identify the degree map $\deg : \mathrm{Pic}_{X/S} \to \mathbb{Z}$ with the connecting map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ and exactness implies $\ker(\deg)$ is the claimed quotient.

2.1 Singular curves

We are interested more generally in the case that $f : X \to S$ is a family of integral but not necessarily smooth curves. Under the usual assumptions, we have proved the existence of the Picard scheme and have constructed an Abel-Jacobi map from an open subscheme of $\mathrm{Hilb}^n_{X/S}$. In the case where the fibers of f are also assumed to be Gorenstein, then the picture is almost identical.

Remark 1. *Recall that a quasi-projective scheme X/k is Gorenstein if it is Cohen-Macaulay and the dualizing sheaf $\omega_{X/k}$ is a line bundle. The most important case for us to note is that local complete intersection varieties, and in particular hypersurfaces, are Gorenstein. In this case, the theory of adjunction tells us that if X is a hypersurface in a smooth variety $\mathbb{P}$, then $\omega_{X/k} = \omega_{\mathbb{P}/k} \otimes \mathcal{O}_{\mathbb{P}}(X)|_X$ where for a smooth variety $\omega_{\mathbb{P}/k} = \Omega^{\dim X}_{X/k}$.*

The main input we need is that Riemann-Roch and Serre duality work as expected for X/k an integral Gorenstein curve.

Theorem 2. *Let X/k be an integral Gorenstein curve with arithmetic genus $g = p_g := \dim H^1(X, \mathcal{O}_X)$. Then for any line bundle L,*

$$H^i(X, L)^\vee \cong H^{1-i}(X, \omega_X \otimes L^\vee)$$

and there exists a number $d = \deg L$ such that

$$\chi(X, L) = \deg L + \chi(\mathcal{O}_X) = \deg L + 1 - g.$$

By the previous remark, the theorem holds in particular whenever the integral curve X is embedded in a smooth surface S. In fact, we ony need to assume such an embedding locally since both the Cohen-Macualay condition and the condition of being a line bundle are local. In particular, we have that curves with singularities that can be embedded in the affine plane[4] are Gorenstein. We call such curves *locally planar*.

[3] More generally, the components $\mathrm{Pic}^n_{X/S}$ are torsors over $\mathrm{Jac}_{X/S}$.

[4] equivalently, have tangent space dimension 2

The upshot, is that for $f : X \to S$ a flat family of projective Gorenstein curves satisfying the usual assumtpions, the structure of $\mathrm{Pic}_{X/S}$ is almost identical to the smooth case. The components are indexed by degree d, the Abel-Jacobi map is a smooth projective bundle above degree $2g-2$ with fibers of dimension $d-g$, and the degree 0 component $\mathrm{Jac}_{X/S} \to S$ is a smooth group scheme of relative dimension g where $g = p_a$ is the arithmetic genus of the family. The one thing that fails is properness, as we have seen.

Let X/k be projective Gorenstein curve with arithmetic genus $g = p_a$ and let $\nu : X^\nu \to X$ be the normalization so that X^ν is a smooth projective curve of genus $p_g \leq g$, the geometric genus of X/k. Pulling back gives us a homomorphism

$$\nu^* : \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X^\nu/k}$$

of group schemes which preserves the degree.

On the other hand, consider the short exact sequence of sheaves on X

$$1 \to \mathcal{O}_X^* \to \nu_* \mathcal{O}_{X^\nu}^* \to \mathcal{F} \to 1$$

where $\mathcal{F}$ is the cokernel of the pullback map on invertible functions. Then $\mathcal{F}$ is a direct sum of skyscraper sheaves of abelian groups supported at the singular points of X. Now we take the long exact sequence of cohomology, noting that $H^1(X, \mathcal{F}) = 0$ since $\mathcal{F}$ is supported on points, that the pullback map on global functions is an isomorphism, and that $H^1(X, \nu_* \mathcal{O}_{X^\nu}^*) = H^1(X^\nu, \mathcal{O}_{X^\nu}^*)$ since ν is finite, we get the short exact

$$1 \to H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X^*) \to H^1(X^\nu, \mathcal{O}_{X^\nu}^*) \to 1$$

of abelian groups. The latter map is the pullback map ν^* on Picard groups and since ν^* preserves degrees, we get a short exact sequence

$$1 \to H^0(X, \mathcal{F}) \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X^\nu) \to 0.$$

The same analysis can be performed for X_T for any $T \to \mathrm{Spec}\, k$ and so we get an exact sequence of group schemes

$$1 \to F \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

where F is the commutative group scheme over k representing the sheafification of the functor

$$T \mapsto H^0(X_T, \mathcal{F}_T)^5$$

The main thing to note is that the group scheme F is a direct sum over each singular point $x \in X$ of a local factor F_x depending only on the stalk $\mathcal{F}_x$ of the skyscraper $\mathcal{F}$. On the other hand, $\mathcal{F}_x$ depends only on the completed local ring $\widehat{\mathcal{O}}_{X,x}$ which allows us to compute F in examples.

Example 1. *(The node) Suppose $x \in X$ has a split nodal singularity. That is, the completed local ring is isomorphic to $R = k[\![x,y]\!]/xy$. The normalization has completed local ring $\tilde{R} = k[\![x]\!] \times k[\![y]\!]$. Then the stalk of $\mathcal{F}_x$ can be computed from the sequence*

$$1 \to R^* \to \tilde{R}^* \to \mathcal{F}_x \to 1.$$

[5] Note since $\mathcal{F}$ is a union of skyscrapers, this is just the locally constant sheaf associated to the group $H^0(X, \mathcal{F})$.

Then the map $\tilde{R}^* \to k^*$ given by $(f,g) \mapsto f(0)/g(0)$ identifies $\mathcal{F}_x$ with k^* so $\mathcal{F}$ is the skyscraper sheaf k^*_x and the group scheme F is simply $\mathbb{G}_m$. More generally, suppose X has exactly δ split nodal singular points and is smooth elsewhere. Then we have an exact sequence

$$1 \to \mathbb{G}_m^{\oplus\delta} \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

where $\mathrm{Jac}_{X^\nu/k}$ is a $g-\delta$ dimensional abelian variety.

Example 2. *(The cusp) Suppose $x \in X$ has a cuspidal singularity with completed local ring isomorphic to $k[\![x,y]\!]/\{y^2 = x^3\}$. Then $\tilde{R} = k[\![t]\!]$ with the map $R \to \tilde{R}$ given by $(x,y) \mapsto (t^3, t^2)$. The cokernel of $R^* \to \tilde{R}^*$ can be identified with the map $\tilde{R} \to k$ given by*

$$g(t) \mapsto \frac{g(t) - g(0)}{t}\Big|_{t=0}.$$

Therefore, $F = \mathbb{G}_a$ is the additive group and we have an exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

3 Compactified Jacobians

Our goal now is to compactify the Jacobian, or more generally Pic^d for $f : X \to S$ a family of locally planar, or more generally Gorenstein, integral curves.

The idea is to again leverage the Abel-Jacobi map as in the construction of $\mathrm{Pic}_{X/S}$. In the case of curves we have the space of degree d Cartier divisors $CDiv^d_{X/S}$ sitting inside of the Hilbert scheme $\mathrm{Hilb}^d_{X/S}$. While $CDiv^d_{X/S}$ is not proper, $\mathrm{Hilb}^d_{X/S}$ is and so the idea is to extend functor of $\mathrm{Pic}_{X/S}$ to something that admits an Abel-Jacobi map from the proper S-scheme $\mathrm{Hilb}^d_{X/S}$ and then construct a representing object as a quotient of $\mathrm{Hilb}^d_{X/S}$ by a flat and proper equivalence relation.

If $D \subset X$ is a length d subscheme that is not necessarily a Cartier divisor, then the ideal sheaf I_D is not a line bundle, but it is a rank 1 torsion free sheaf.

Definition 3. *Let X/k be an integral variety over a field. A torsion free sheaf on X is a coherent sheaf $\mathcal{E}$ such that the support* $\mathrm{Supp}(\mathcal{E})$ *has no embedded points. Equivalently, the annihilator of $\mathcal{E}$ is the* 0 *ideal. The rank of a torsion free sheaf is the rank of the generic fiber $\mathcal{E}_\eta$.*

Now to see that I_D for $D \subset X$ a closed subscheme of an integral curve X/k is a rank 1 torsion free sheaf, note that $I_D \subset \mathcal{O}_X$ and $\mathcal{O}_X$ is torsion free. Moreover, the inclusion is an isomorphism away from D so the rank of I_D is 1. In this setting, a point $D \subset X$ of the Hilbert scheme is called a generalized divisor.

Definition 4. *The degree of a rank* 1 *torsion free sheaf I on an integral curve X/k is defined as*

$$\chi(I) - \chi(\mathcal{O}_X).$$

This definition of degree generalizes the degree of a line bundle on a smooth projective curve as computed by Riemann-Roch. Now we can define the compactified Picard functor.

Definition 5. *Let $f : X \to S$ be a projective family of integral curves. A family of rank* 1 *degree d torsion free sheaves on X is an S-flat coherent sheaf $\mathcal{I}$ on X such that $\mathcal{I}|_{X_s}$ is a rank* 1 *degree d torsion free sheaf on X_s for each $s \in S$.*

Definition 6. *Let $f : X \to S$ be a flat projective family of integral curves. For each integer d, the compactified Picard functor $\overline{Pic}^d_{X/S} : Sch_S \to Set$ given by*

$$T \mapsto \{\textit{families of rank } 1 \textit{ degree } d \textit{ coherent sheaves on } X_T \to T\}/\mathrm{Pic}(T).$$

Note that a line bundle L on X_T is in particular a family of rank 1 degree d coherent sheaves on X_T so that $\mathrm{Pic}^d_{X/S}$ is a subfunctor of $\overline{\mathrm{Pic}}^d_{X/S}$. The special case $d = 0$, the compactified Jacobian, will be denoted by $\overline{\mathrm{Jac}}_{X/S}$.

Remark 2. *Note that as in the case of the usual Picard functor, if our family of $f : X \to S$ has a section σ that is contained in the regular locus, then $\sigma(S)$ is a relative Cartier divisor of degree 1 and twisting by $\mathcal{O}_X(-d\sigma(S))$ gives an isomorphism of functors*

$$\overline{\mathrm{Pic}}^d_{X/S} \to \overline{\mathrm{Jac}}_{X/S}.$$

This happens in particular if $S = \mathrm{Spec}\, k$ and $X \setminus X^{sing}$ has a rational point.

The idea now is to extend the Abel-Jacobi map $AJ^d_{X/S} : CDiv^d_{X/S} \to \mathrm{Pic}^d_{X/S}$ to the compactified Jacobian.

$$\begin{array}{ccc} \mathrm{Hilb}^d_{X/S} & \xrightarrow{AJ^d_{X/S}} & \overline{\mathrm{Pic}}^d_{X/S} \\ \uparrow & & \uparrow \\ CDiv^d_{X/S} & \xrightarrow[AJ^d_{X/S}]{} & \mathrm{Pic}^d_{X/S} \end{array}$$

The extension must be defined on points by sending a flat closed subscheme of degree d $D \subset X_T$ to rank 1 torsion free sheaf $I_D^\vee := \mathcal{H}om_{X_T}(I_D, \mathcal{O}_{X_T})$. The problem is that when I_D is not a line bundle, taking duals isn't well behaved in general so its not clear this is a well defined natural transformation of functors. However, we have the following results due to Hartshorne. Let us denote $I_D^\vee$ by $\mathcal{O}_X(D)$ in analogy with the Cartier case.

Proposition 5. *(Properties of generalized divisors on Gorenstein curves) Suppose X/k is a Gorenstein integral curve and let I be a rank 1 torsion free sheaf on X. We have the following:*

(a) the natural map $I \to (I^\vee)^\vee$ is an isomorphism[6]*,*

(b) $\deg \mathcal{O}_X(D) = \deg D$*,*

(c) Riemann-Roch and Serre duality hold for $\mathcal{O}_X(D)$.

Moreover, we have the usual correspondence between the set of $D \subset X$ such that $\mathcal{O}_X(D) \cong L$ and sections $H^0(X, L)$. Note that these two are also in bijection with $\mathrm{Hom}_X(I_D, \mathcal{O}_X)$.

The above facts tell us that the Abel-Jacobi map for $\overline{\mathrm{Pic}}^d_{X/S}$ works as expected on the level of points when $f : X \to S$ is a flat projective family of integral Gorenstein curves. The missing piece is that it behaves well under base-change. This comes from a certain generalization of the cohomology and base change theorem for $\mathcal{E}xt^i$ groups rather than cohomology groups due to Altman and Kleiman. In this particular case we get the following.

[6]that is, I is a reflexive sheaf

Theorem 3. *(Altman-Kleiman) Suppose $f : X \to S$ is a flat projective family of integral Gorenstein curves and I a family of torsion free sheaves on $f : X \to S$. Then $\mathcal{H}om_X(I, \mathcal{O}_X)$ is flat and its formation commutes with arbitrary base change.*

The key point here is that the vanishing of H^1 that implies that pushforwards commute with basechange is replaced in this case with a vanishing $\mathrm{Ext}^1_{X_k}(I_k, \mathcal{O}_{X_k}) = 0$ which holds since X_k is Gorenstein. This implies that the Abel-Jacobi map is a well defined natural transformation of functors, and by repeating the argument for the Picard group we obtain the following theorem of Altman and Kleiman.

Theorem 4. *Let $f : X \to S$ be a flat projective family of integral Gorenstein curves over a Noetherian scheme satisfying conditions $(**)$. Then $\overline{\mathrm{Pic}}^d_{X/S}$ is representable by a projective S-scheme. Moreover, the Abel-Jacobi map*

$$AJ^d_{X/S} : \mathrm{Hilb}^d_{X/S} \to \overline{\mathrm{Pic}}^d_{X/S}$$

is identified with the projectivization of a coherent sheaf. When $d > 2g - 2$ where g is the arithmetic genus of $f : X \to S$, the Abel-Jacobi map is a smooth projective bundle of rank $d - g$.

Example 3. *Let X/k be a projective geometrically integral Gorenstein curve of genus 1. Then the the $d = 1$ Abel-Jacobi map $\mathrm{Hilb}^1_{X/k} = X \to \overline{\mathrm{Pic}}^1_{X/k}$ is a smooth projective bundle of rank $1 - 1 = 0$, that is, its an isomorphism. In this case, $\overline{\mathrm{Pic}}^1_{X/k}$ is the curve itself and the points in the boundary*

$$\overline{\mathrm{Pic}}^1_{X/k} \setminus \mathrm{Pic}^1_{X/k}$$

correspond to the maximal ideal I of the singular point, or more precisely, its $\mathcal{O}_X$ dual $\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_X)$.

4 The topology of compactified Jacobians

For this section let us work over $k = \mathbb{C}$ the complex numbers. Let X/k be a projective integral Gorenstein curve. To study the topology of $\overline{\mathrm{Jac}}_X$, we will leverage the action of Jac_X by tensoring with a degree 0 line bundle.

Toward that end, let $\mathcal{I}$ a rank 1 degree 0 torsion free sheaf and L a line bundle. Consider the endormorphism algebra $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{I})$. This is a finite extension of $\mathcal{O}_X$ with generic fiber equal to the function field $k(X)$. Thus $X' := \mathrm{Spec}\,_X\mathcal{A}$ is an integral curve mapping finitely and birationally to X. That is, $f : X' \to X$ is a partial normalization of X. Moreover, $\mathcal{I}$ is an $\mathcal{A}$-algebra and by construction, $f_*\mathcal{O}_{X'} = \mathcal{A}$, therefore $\mathcal{I}$ is an $f_*\mathcal{O}_{X'}$-module and by pulling back sections we get an $\mathcal{O}_{X'}$-module $\mathcal{I}'$ such that $f_*\mathcal{I}' = \mathcal{I}$. In this way, every rank 1 torsion free sheaf on X is pushed forward from some partial normalization.

Lemma 1. *The sheaf $\mathcal{I} \otimes L$ is isomorphic to $\mathcal{I}$ if and only if $f^*L \cong \mathcal{O}_{X'}$ where $f : X' \to X$ is the partial normalization associated to $\mathcal{I}$.*

Proof. If f^*L is trivial, then $\mathcal{I} \otimes L = f_*(\mathcal{I}' \otimes f^*L) = f_*\mathcal{I}' = \mathcal{I}$ by the projection formula. Similarly, consider

$$\mathcal{H}om(\mathcal{I}, \mathcal{I} \otimes L) = \mathcal{E}nd(\mathcal{I}) \otimes L = f_*\mathcal{O}_{X'} \otimes L = f_*(\mathcal{O}_{X'} \otimes f^*L) = f_*f^*L.$$

Then if $\mathcal{I} \cong \mathcal{I} \otimes L$, such an isomorphism would give a nonzero section of f^*L. On the other hand, f^*L is a degree 0 line bundle so if it has a section it is trivial. □

We will consider the topological Euler characteristic e_{top} of $\overline{\mathrm{Pic}}^d_X$. This is a topological invariant valued in the integers. We will need the following properties of the Euler characteristic.

Fact 1. *1. If $Z \subset X$ is a closed subvariety and $X \setminus Z = U$ the open complement, then $e_{top}(X) = e_{top}(U) + e_{top}(Z)$.*

2. If $f : X \to Y$ smooth and proper morphism then $e_{top}(X) = e_{top}(Y)e_{top}(F)$ where F is any fiber of f.[7] More generally, suppose f is a proper fibration, the same is true.

3. $e_{top}(point) = 1$ and $e_{top}(S^1) = 0$. In particular $e_{top}(torus) = 0$.

When X is smooth, then the Jacobian is a $g(X)$ dimensional abelian variety. In particular, by the third fact, $e_{top}(\mathrm{Jac}_X) = 1$ if $g = 0$ since Jac_X is a point, and $e_{top}(\mathrm{Jac}_X) = 0$ for $g > 0$ since it is topologically a torus. That is, one can distinguish smooth rational curves by $e_{top}(\mathrm{Jac}_X)$. The following proposition generalizes this.

Proposition 6. *Suppose the normalization X^ν of X has genus $g(X^\nu) \geq 1$. Then $e_{top}(\overline{\mathrm{Jac}}_X) = 0$.*

Proof. Consider the exact sequence of group schemes

$$0 \to F \to \mathrm{Jac}_X \to \mathrm{Jac}_{X^\nu} \to 0.$$

We saw above that F is an extension of multiplicative and additive groups $\mathbb{G}_m$ and $\mathbb{G}_a$. In particular, F is divisible as an abstract abelian groups and so this sequence splits as a sequence of abelian groups[8]. Since $g(X^\nu) \geq 1$, then Jac_{X^ν} is an abelian variety of dimension at least 1. Thus for each n, there exists an element of order n. Using this noncanonical splitting, we can lift this to an element of order in Jac_X, that is, a line bundle L on X with $L^{\otimes n} \cong \mathcal{O}_X$. Then the pullback of L to X^ν is nontrivial by construction since it pulls back to an element of order n. In particular, for any partial normalization $f : X' \to X$, f^*L is nontrivial. Thus by the previous lemma, for any rank 1 torsion free sheaf $\mathcal{I}$ on X, $\mathcal{I} \otimes L \ncong \mathcal{I}$. That is, the action of L has no fixed points on $\overline{\mathrm{Jac}}_X$. In fact tensoring by L induces a free action of $\mathbb{Z}/n\mathbb{Z}$ on $\overline{\mathrm{Jac}}_X$. Therefore $e_{top}(\overline{\mathrm{Jac}}_X)$ is divisible by n, but n was arbitrary so $e_{top}(\overline{\mathrm{Jac}}_X) = 0$. □

Now let $f : \mathcal{X} \to S$ be some flat and propre family of integral Gorenstein curves over $\mathbb{C}$ and suppose $S_{rat} := \{s \in S \mid \mathcal{X}_s \text{ is rational}\}$[9] is a finite set. Then we have the relative compactified Jacobian

$$\overline{\mathrm{Jac}}_{\mathcal{X}/S} \to S$$

which is proper over S. Now for any proper map $Y \to S$ of complex varieties, there is a locally closed decomposition $S = \sqcup S_\alpha$ such that $Y_\alpha \to S_\alpha$ is a proper fibration with fiber F_α. Using additivity and multiplicativity propertes of e_{top} above, we see that

$$e_{top}(Y) = \sum_\alpha e_{top}(S_\alpha)e_{top}(F_\alpha).$$

In our case at hand where $Y = \overline{\mathrm{Jac}}_{\mathcal{X}/S} \to S$, by the proposition, we see that $e_{top}(F_\alpha) = 0$ over any stratum where where the curve $\mathcal{X}_s$ has geometric genus ≥ 1. Therefore the whole sum collapses to the points S_{rat} which are assumed to be finite. Therefore we get the following computation.

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[7]Note that the fibers of a smooth and proper morphism have diffeomorphic underlying complex manifolds.

[8]not necessarily as group schemes

[9]Recall a curve is rational of the genus of X^ν is 0, that is, $X^\nu \cong \mathbb{P}^1$.

Proposition 7.

$$e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}/S}) = \sum_{s \in S_{rat}} e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}_s}).$$

In particular, $e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}/S})$ counts the number of rational curves in the fibers $f : \mathcal{X} \to S$, weighted with multiplicity given by the topological Euler characteristic of their compactified Jacobian. Beauville used this to give a proof of a remarkable formula of Yau-Zaslow counting the number of rational curves on a K3 surface which we will now sketch.

5 The Yau-Zaslow formula

We continue working over $\mathbb{C}$. Recall that a *K3 surface* is a smooth projective surface X with trivial canonical sheaf

$$\omega_X := \Lambda^2 \Omega_X \cong \mathcal{O}_X$$

and $H^1(X, \mathcal{O}_X) = 0$. A *polarized* K3 surface is a pair (X, H) where X is a K3 surface and H is an ample line bundle. The degree of (X, H) is $d = c_1(H)^2$.

Consider the linear series $|H| = \mathbb{P}(H^0(X, H))$. It is a g-dimensional space where $d = 2g - 2$ and the curves in $|H|$ have arithmetic genus g by the adjunction formula. Inside $X \times |H|$ we have a universal family of curves $\mathcal{C} \to |H|$ with the fiber over a point being the curve in the linear series parametrized by that point. Indeed one can identify $|H|$ with a component of the Hilbert scheme corresponding to effective Cartier divisors D with $\mathcal{O}_X(D) \cong H$. In general, this is an Abel-Jacobi fiber but in this case we see that $H^1(X, \mathcal{O}_X) = 0$ by definition of a K3 so Pic_X is zero dimensional and so the fibers are the components. Then $\mathcal{C} \to |H|$ is simply the universal family of the Hilbert scheme over this component.

Lemma 2. *There are finitely many rational curves parametrized by $|H|$.*

Proof. Suppose that the locus in $|H| \cong \mathbb{P}^g$ parametrizing rational curves is higher dimensional. Then there exists an irreducible curve $B_0 \subset |H|$ contained in the rational locus and over B_0 there is a family of rational curves $R_0 \to B_0$. Taking the normalization of both sides, we obtain a family $R \to B$ where B is integral and the generic fibers are smooth rational curves. Thus $R \to B$ contains a generically ruled surface $R' \to B$ as a component. On the other hand, we have a map $R' \to X$ which is dominant. This is a contradiction as X is a K3. □

Let $n(g)$ denote the number of rational curves in $|H|$ for a generic polarized complex K3 surface (X, H) Then we have the following formula of Yau-Zaslow. We will sketch the proof of Beauville which is based on the topology of compactified Jacobians.

Theorem 5 (Yau-Zaslow)**.**

$$1 + \sum_{g \geq 1} n(g) q^g = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}}$$

In particular, the numbers $n(g)$ are constant for general (X, H).

Proof. (Sketch) Let (X, H) be a generic genus g K3 surface. It is hard theorem which we won't cover here that in this case, every curve in $|H|$ is integral and in fact has at worst nodal singularities.[10] Let $\mathcal{C} \to |H|$ be the universal family of curves in this linear series. By the integrality we have a relative compactified Picard. consider the degree g piece

$$\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|} \to |H|.$$

This family is fiberwise isomorphic to $\overline{\mathrm{Jac}}_{\mathcal{C}/|H|} \to |H|$ (though it could be globally different if there is no global section of $\mathcal{C} \to |H|$) so the argument in the previous section tells us that

$$e_{top}(\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}) = \sum_{\mathcal{C}_s \text{ rational curves in } |H|} e_{top}(\overline{\mathrm{Jac}}_{\mathcal{C}_s}).$$

We saw in the above examples that C is a nodal cubic, $\overline{\mathrm{Jac}}_C \cong C$ and in particular has topological euler characteristic 1. This generalizes as follows.

Lemma 3. *If C is an integral rational curve with at worst nodal singularities, then $e_{top}(\overline{\mathrm{Jac}}_C) = 1$.*

We won't give the details of the proof but the idea is that topologically, $\overline{\mathrm{Jac}}_C$ is a product over local contributions that are each homeomorphic to the above example and so the euler characteristic is still 1. Thus we get that

$$n_g = e_{top}(\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}).$$

Now given a point of $\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}$ corresponding to a pair (C, L) where C is smooth and L is a line bundle of degree g, then

$$\chi(C, L) = 1$$

by Riemann-Roch. On the other hand, by semi-continuity of coherent cohomology, there exists an open subset $U \subset \overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}$ parametrizing such pairs where $H^1(C, L) = 0$. On this open subset, we in fact that that $H^0(C, L) = 1$ and so L has a unique section. The zero locus of this section is a zero dimensional degree g subscheme of X which gives a point of Hilb^g_X, the Hilbert scheme of g points. This gives a rational and generically injective map

$$\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|} \dashrightarrow \mathrm{Hilb}^g_X.$$

The source and the target of this map are in fact smooth holomorphic symplectic varieties[11] of the same dimension. In particular, this map is birational and then it follows from a result of Batyrev and Kontsevich or a result of Huybrechts that the source and the target then have the same euler characteristics.[12]

Thus we have that

$$n_g = e_{top}(\mathrm{Hilb}^g_X).$$

[10]Of course this isn't true for any (X, H) and here is where the genericity assumption comes in. In fact this statement was only conjectured at the time of the proof of the Yau-Zaslow formula and it was only proved a few years later.

[11]A holomorphic symplectic variety V is one with a holomorphic 2-form $\omega H^0(V, \Omega^2_V)$ which is anti-symmetric, closed, and nondegenerate. In this case the existence of such a form on these two moduli spaces follows from more general work of Mukai on moduli of sheaves on a K3 surface, which both of these spaces are examples of.

[12]In fact they are even diffeomorphic.

Finally, in the next few classes we will study the geometry of the Hilbert scheme of points on surfaces and prove both the above smoothness and irreducibility claim, as well as the formula that in this particular case of X being a K3,

$$\sum_{g \geq 0} e_{top}(\mathrm{Hilb}^g_X) q^g = \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}},$$

completing the proof. □