Lecture 2: Moduli functors and Grassmannians

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1 Moduli functors and representability

We arrive at the precise definitions that form the backbone of moduli theory.

Let $\mathcal{C}$ be any category. Given an object X of $\mathcal{C}$, we can consider the (contravariant) functor of points associated to X:

$$h_X : \mathcal{C}^{op} \to Set \tag{1}$$
$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X) \tag{2}$$

Note that $h\_$ defines a covariant functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, Set)$: if $a : X \to Y$ is a morphism, then $h_a : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$ is given by composition with a.

We have the following basic but crucial lemma.

Lemma 1 (Yoneda)**.** *(a) For any object X of $\mathcal{C}$ and any functor $F : \mathcal{C} \to Set$, there is a natural isomorphism*

$$\mathrm{Nat}(h_X, F) \cong F(X).$$

(b) The functor

$$h\_ : \mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$

is fully faithful.

In light of this, we will often view $\mathcal{C}$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, Set)$ and identify objects X of $\mathcal{C}$ with their functor of points h_X. We often call a functor F a presheaf on $\mathcal{C}$ and refer to $Fun(\mathcal{C}^{op}, Set)$ as the category of presheaves.

Definition 1. *We say that a functor $F : \mathcal{C}^{op} \to Set$ is representable if there exists an object X and a natural isomorphism*

$$\xi : F \to h_X.$$

In this case we say F is representable by X.

In this course, the case of interest is when $\mathcal{C} = Sch_S$ is the category of schemes over a base S. In this case, we call F a *moduli problem* or *moduli functor*. In most cases F will be of the form

$$F(T) = \{\text{families of objects over } T\}/\sim$$

where $\sim$ is isomorphism, and F is made into a functor by pulling back families along $T' \to T$.

Definition 2. *If F is representable by a scheme M, we say that M is a fine moduli space for the moduli problem F.*

Given a fine moduli space M for F, which is unique if it exists, then we have an element $\xi^{-1}(id_M) \in F(M)$ corresponding to the identity $\mathrm{Hom}_{\mathcal{C}}(M, M)$. In the above picture, this corresponds to a family $U \to M$ over M for the moduli problem F. By (a slightly stronger version of) Yoneda's lemma, this family has the following strong universal property: for any base scheme T and any family $U_T \to T$ in $F(T)$, there exists a morphism $T \to M$ and a pullback square

$$\begin{CD} U_T @>>> U \\ @VVV @VVV \\ T @>>> M. \end{CD}$$

When $T = \mathrm{Spec}\ k$, we get a bijection between the set of isomorphism classes of objects over k and the k points $M(k)$ of our moduli space. More generally, we have a bijection between families of objects over T and morphisms $T \to M$ given by pulling back the universal family $U \to M$. In some sense, all the geometry of all families of objects at hand are captured by the geometry of the universal family $U \to M$ over the moduli space M.

Often fine moduli spaces don't exist, but we have the following slightly weaker notion.

Definition 3. *A scheme M and a natural transformation $\xi : F \to h_M$ is a coarse moduli space if*

(a) $\xi(k) : F(\mathrm{Spec}\ k) \to h_M(\mathrm{Spec}\ k)$ is a bijection for all algebraically closed fields k, and

(b) for any scheme M' and any natural transformation $\xi' : F \to h_{M'}$, there exists a unique morphism $\alpha : M \to M'$ such that ξ' factors as $h_\alpha \circ \xi$.

We can think of a coarse moduli space as the initial scheme whose closed points correspond to objects of our moduli problem. However, coarse moduli spaces need not have universal families. It is clear from the definition that a fine moduli space for F is a coarse moduli space.

Example 1. *The global sections functor $Sch_S \to Set$ given by $X \mapsto \mathcal{O}_X(X)$ is representable by $\mathbb{A}^1_S$. The universal global section is $x \in \mathcal{O}_S[x]$ where $\mathbb{A}^1_S = \mathrm{Spec}_S\ \mathcal{O}_S[x]$.*

Example 2. *The scheme $\mathbb{P}^n_S$ represents the following functor on the category Sch_S.*

$$X \mapsto \{(L, s_0, \ldots, s_n)\ : \textit{satisfying condition (*)}\} / \sim .$$

Here L is a line bundle, $s_i \in H^0(X, L)$ are global sections of L, and condition $(*)$ is that for each $x \in X$, there exists an i such that $s_i(x) \neq 0$. Two such data $(L, s_0, \ldots, s_n)$ and $(L', s'_0, \ldots, s'_n)$ are equivalent if there exists an isomorphism of line bundles

$$\alpha : L \to L'$$

with $\alpha(s_i) = s'_i$. Here the universal line bundle with sections on $\mathbb{P}^n$ is given by $(\mathcal{O}_{\mathbb{P}^n}(1), x_0, \ldots, x_n)$. Another way to write condition $(*)$ is that the map of sheaves

$$\mathcal{O}_X^{n+1} \to L$$

induced by the s_i is surjective.

1.1 Criteria for representability

Recall that a presheaf F on Sch_S is a (Zariski) *sheaf* if for any X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer.

$$F(X) \to \prod_i F(U_i) \rightrightarrows F(U_i \cap U_j)$$

Proposition 1. *Representable functors are sheaves for the Zariski topology.*

Proof. We need to check that for any scheme X, $h_X = \mathrm{Hom}_S(-, X)$ is a sheaf. This follows from the fact that we can glue morphisms. □

This gives us our first criterion for ruling out representability of a functor. In particular, given a candidate moduli functor, it had better sheafify it to have any hope of representable.[1]

The following is a useful property of the category of presheaves.

Lemma 2. *The category $Fun(\mathcal{C}^{op}, Set)$ is closed under limits and colimits. Furthermore, the Yoneda functor $h\_$ preserves limits.*[2]

Definition 4. *(a) We say that a subfunctor F of a functor G is open (respectively closed) if and only if for any scheme T and any morphism $T \to G$, the pullback $T \times_G F$ is representable by an open (respectively closed) subscheme of T.*

(b) We say that a collection of open subfunctors F_i of F is an open cover of F if for any scheme T and any morphism $T \to F$, the pullbacks $\{U_i := T \times_F F_i \to T\}$ form an open cover of T.

We can rephrase the above definitions using the moduli functor language as follows. An open (resp. closed) subfunctor $F \subset G$ is one such that for any family $\xi \in G(T)$, there is an open set $U \subset T$ (resp. closed subset $Z \subset T$) such that a morphism $f : T' \to T$ factors through U (resp. Z) if and only if $f^*\xi \in F(T')$. Similarly, a collection of open subfunctors $\{F_i \subset F\}$ form an open cover if for any $\xi \in F(T)$, there exists an open cover $\{U_i \to T\}$ such that $\xi|_{U_i} \in F_i(U_i)$.

Proposition 2. *Let $F \in Fun(Sch_S^{op}, Set)$ be a functor such that*

(a) F is a Zariski sheaf, and

(b) F has an open cover $\{F_i\}$ by representable open subfunctors.

Then F is representable by a scheme.

Proof. Let X_i be the scheme representing F_i with universal object $\xi_i \in F_i(X_i)$. We can consider the pullback

$$\begin{array}{ccc} U_{ij} & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ F_j & \longrightarrow & F \end{array}$$

[1] See [1, II.1.2] for sheafification.

[2] Note it does not in general preserve colimits.

where $U_{ij} \subset X_i$ is an open immersion since F_j is an open subfunctor. Furthermore, we have an equality

$$\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$$

by commutativity of the pullback diagram. This induces an isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

$$\varphi_{ij}^* \xi_j = \xi_i|_{U_{ij}}.$$

Now we want to construct a scheme X along with an object $\xi \in F(X)$ by gluing the schemes X_i along the open subsets U_{ij} using the isomorphisms φ_{ij}. We need to check that the isomorphisms $\varphi_{ij} : U_{ij} \cong U_{ji}$ satisfy the cocycle condition. To make sense of this, we first want to know that φ_{ij} identifies

$$U_{ij} \cap U_{ik} \cong U_{ji} \cap U_{jk}.$$

This follows since the left side (resp. the right side) is characterized by the fact that $\xi_i|_{U_{ij} \cap U_{ik}} \in F_k(U_{ij} \cap U_{ik})$ (resp. $\xi_j|_{U_{ji} \cap U_{jk}} \in F_k(U_{ji} \cap U_{jk})$) and $\varphi_{ij}^* \xi_j = \xi_i$.

Now it makes sense to require that

$$\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}} = \varphi_{ik}|_{U_{ik} \cap U_{ij}}$$

as maps $U_{ij} \cap U_{ik} \to U_{ki} \cap U_{kj}$. This follows since both maps pullback ξ_k to ξ_i.

Now we can glue the X_i along the open subsets U_{ij} using the isomorphisms φ_{ij} to obtain a scheme X. Moreover, the universal objects ξ_i over X_i are identified on the overlaps U_{ij} and so since F is a Zariski sheaf, the ξ_i glue to form a $\xi \in F(X)$ induced by a morphism $X \to F$.

Now we need to show that (X, ξ) represents F. Let T be a scheme with a morphism $T \to F$ induced by an object $\zeta \in F(T)$. Since F_i form an open cover, there exists an open cover U_i of T such that $\zeta|_{U_i} =: \zeta_i$ defines a morphism $U_i \to X_i$. Moreover,

$$\zeta_i|_{U_i \cap U_j} = \zeta_j|_{U_i \cap U_j}$$

so the morphisms $U_i \to X_i$ glue to give a morphism $f : T \to X$ with $f^* \xi = \zeta$.

□

2 Grassmannians

Definition 5. *For any k, n, let $Gr(k, n)$ denote the functor $Sch \to Set$ given by*

$$S \mapsto \{\alpha : \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{V}\} / \sim$$

where α is a surjection, $\mathcal{V}$ is a rank k locally free sheaf, and $\sim$ is given by isomorphism $\mathcal{E} \cong \mathcal{E}'$ commuting with the surjections α and α'.

We will use the above representability criteria to construct a scheme representing $Gr(k, n)$. Note that when $k = 1$, we recover the functor represented by $\mathbb{P}^{n-1}$ as above.

Remark 1. *Let $\mathcal{E} = \ker(\alpha : \mathcal{O}_S^{\oplus n} \to \mathcal{V})$. Since $\mathcal{V}$ is locally free and the sequence*

$$0 \to \mathcal{E} \to \mathcal{O}_S^{\oplus n} \to 0$$

is exact, then for each $x \in S$, we have

$$0 \to \mathcal{E}|_x \to k(x)^n \to \mathcal{V}_x \to 0$$

is exact. Thus $\mathcal{E}|_x$ is an $n-k$-dimensional subspace of $k(x)^n$ for each $x \in S$. In particular $\mathcal{E}$ is a rank $n-k$ locally free sheaf on S and we can think of the inclusion of $\mathcal{E} \to \mathcal{O}_S \otimes V$ as a family of rank $n-k$ subspaces of an n-dimensional vector space parametrized by the scheme S. More precisely, this identifies the Grassmannian functor with the functor

$$S \mapsto \{\textit{rank } n-k \textit{ sub-bundles of } \mathcal{O}_S^n\}.$$

Let us give some a sketch of the construction over a field that we will make more precise later. When S is the spectrum of an algebraically closed field, $\mathcal{V}$ is just the trivial bundle and so a map $\alpha : \mathcal{O}_S^{\oplus n} \to \mathcal{O}_S^{\oplus k}$ is given by a $k \times n$ matrix. The condition that α is surjective is that the $k \times k$ minors don't all vanish. Finally, isomorphism is given by the action of GL_k on the left.

Thus, set theoretically, the set of closed points of the Grassmannian is the quotient set U/GL_k where $U \subset \mathbb{A}^{nk}$ is the open subset of the space of $k \times n$ matrices of full rank.

To give it the structure of a variety over a field, we note that for each subset $i \subset \{1, \ldots, n\}$ of size k, we can consider the set of full rank $k \times n$ matrices where the i^{th} minor doesn't vanish. Then using the GL_k action, put such a matrix into a form where the i^{th} minor is the identity matrix. E.g. if $i = \{1, \ldots, k\}$ then we act by GL_k so our matrix looks like

$$\begin{bmatrix} 1 & & & & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n} \\ & 1 & & & \vdots & \ddots & & \vdots \\ & & \ddots & & \vdots & & \ddots & \vdots \\ & & & 1 & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n} \end{bmatrix}$$

This identifies GL_k orbits of such matrices with an affine space $\mathbb{A}^{k(n-k)}$ and we can glue these affine spaces together by changing basis. This gives $Gr(k,n)$ the structure of an affine variety.

We will upgrade the above construction to obtain a proof over a general base scheme using the above representability criterion.

Theorem 1. *$Gr(k,n)$ is representable by a finite type scheme over* $\operatorname{Spec} \mathbb{Z}$.

We will prove this next time using the representability criterion above. For each subset $i \subset \{1, \ldots, n\}$ of size k, we will define a subfunctor F_i of the Grassmannian functor as follows. First, let

$$s_i : \mathcal{O}_S^k \to \mathcal{O}_S^n$$

denote the the inclusion where the j^{th} direct summand is mapped by the identity to the i_j^{th} direct summand. Now let F_i be defined as the subfunctor

$$F_i(S) = \{\alpha : \mathcal{O}_S^n \to \mathcal{V} \mid \alpha \circ s_i \text{ is surjective } \} \subset Gr(k,n)(S).$$

References

[1] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157