# Lecture 22: Hilbert schemes of points on surfaces

## 1 The topology of Hilbert schemes of points on surfaces

Let $S = \mathbb{C}$ and $X/S$ a smooth quasi-projective surface. Our goal now is to study the topology of the Hilbert schemes of points $\mathrm{Hilb}^n_X$ parametrizing subschemes $Z \subset X$ with Hilbert polynomial constant $n$. That is, $Z$ is a zero dimensional subscheme with

$$\dim_{\mathbb{C}} \mathcal{O}_Z = n.$$

Our goal is to sketch the proof of the following theorem, which is a combination of results due Fogarty, Briançon, and Göttsche.

**Theorem 1.** *Let $X/\mathbb{C}$ be a smooth quasi-projective surface. Then $\mathrm{Hilb}^n_X$ is a smooth and irreducible quasi-projective $2n$-fold. Moreover, the topological Euler characteristic of the Hilbert schemes of points on $X$ are given by the following formula.*

$$\sum_{n\geq 0} e_{top}(\mathrm{Hilb}^n_X)q^n = \prod_{m\geq 0} \frac{1}{(1-q^n)^{e_{top}(X)}}$$

This completes the sketch of the proof of the Yau-Zaslow formula from last class, and in fact also implies the following about compactified Jacobians.

**Theorem 2.** *Let $C$ be an integral locally planar[1] curve over $\mathbb{C}$. Then $\overline{\mathrm{Jac}}_C$ is an irreducible variety of dimension $g = g(C)$. In particular, $\mathrm{Jac}_C \subset \overline{\mathrm{Jac}}_C$ is dense.*

## 2 The case of $X = \mathbb{A}^2$

The Hilbert scheme $\mathrm{Hilb}^n_{\mathbb{A}^2}$ admits a particularly concrete combinatorial description due to Haiman. Let us denote $\mathrm{Hilb}^n_{\mathbb{A}^2}$ by $\mathbf{H}^n$. Since $\mathbb{A}^2$ is affine, we can identify $\mathbf{H}^n$ with the set

$$\{I \subset \mathbb{C}[x,y] \mid \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}.$$

### 2.1 The torus action

There is an action of the algebraic torus $T = \mathbb{G}^2_{m,t_1,t_2}$ on $\mathbb{A}^2_{x,y}$ which on polynomial functions is given

$$f(x,y) \mapsto f(t_1x, t_2y).$$

[1]That is, the tangent space dimension is at most 2 at each $p \in C$.

This extends to an action on $\mathbf{H}^n$ by

$$t \cdot I = \{f(t_1 x, t_2 y) \mid f \in I\}.$$

The torus fixed points, denoted by $(\mathbf{H}^n)^T$, correspond to those ideals generated by $f$ such that $f(t_1 x, t_2 y) = t_1^a t_2^b f(x,y)$, that is, by monomials $x^a y^b$.

Recall that a *partition of* $n$, $\lambda \vdash n$, is a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ such that

$$\sum \lambda_i = n.$$

The number $k = l(\lambda)$ is the *length* of $\lambda$, the size $n$ is denoted by $|\lambda|$, and the $\lambda_i$ are the *parts* of $\lambda$. We can represent a partition by its *Young diagram*, a left aligned arrangement of boxes with $\lambda_j$ boxes in the $j^{th}$ row. We identify $\lambda$ with its Young diagram and label $\lambda$ as a subset of $\mathbb{N}^2$ with the box in the $i^{th}$ column and $j^{th}$ row labeled by $(i,j)$.

**Lemma 1.** *There is a bijection between monomial ideals in* $\mathbf{H}^n$ *and partitions of* $n$ *given by*

$$I \mapsto \lambda(I) = \{(i,j) \mid x^i y^j \notin I\}$$

*and*

$$\lambda \mapsto I_\lambda = (\{x^r y^s \mid (r,s) \notin \lambda\}.$$

This is enough to compute the Euler characteristic of $\mathbf{H}^n$ by the following fact. Let $Y$ be a finite type $\mathbb{C}$-scheme with an action of an algebraic torus $T = \mathbb{G}_m^r$ and let $Y^T$ be the $T$-fixed locus. Then

$$e_{top}(Y) = e_{top}(Y^T).$$

Indeed $e_{top}(T) = 0$ for any $r > 0$ and non-zero dimensional orbit of $T$ is homeomorphic to an algebraic torus and so only the set of zero dimensional orbits, i.e., $Y^T$, contributes to the Euler characteristic. As a corollary, we obtain

**Corollary 1.** *The Euler characteristic* $e_{top}(\mathbf{H}^n) = p(n)$ *the number of partitions of* $n$*. Moreover,*

$$\sum_{n \geq 0} e_{top}(\mathbf{H}^n) q^n = \prod_{m \geq 1} \frac{1}{1 - q^m}.$$

*Proof.* By the previous fact,

$$e_{top}(\mathbf{H}^n) = e_{top}((\mathbf{H}^n)^T)$$

by $(\mathbf{H}^n)^T$ is the finite set of monomial ideals so its Euler characteristic is just the cardinality. Thus,

$$e_{top}(\mathbf{H}^n) = \#\{\text{monomial partitions}\} = \#\{\lambda \vdash n\}.$$

Thus, it suffices to compute the generating series

$$\sum_{n \geq 0} p(n) q^n.$$

Consider the infinite product

$$\prod_{m \geq 1} \frac{1}{1 - q^m}$$

and expand using $1/(1-x) = \sum x^i$. Given a partition $\lambda$, we can write it as

$$\sum m k_m = n$$

where there are are $k_m$ parts of size $m$. Then the infinite product exactly counts expressions of this form. $\square$

## 2.2 Local structure

Let $B_\lambda = \{x^i y^j \mid (i,j) \in \lambda\}$. Note that $B_\lambda$ forms a basis for $\mathbb{C}[x,y]/I_\lambda$. We will define open subfunctors $U_\lambda \subset \mathbf{H}^n$ as follows. Given any test scheme $S$ and a map $S \to \mathbf{H}^n$ corresponding to a closed subscheme $Z \subset S \times \mathbb{A}^2$ flat over $S$, the pushforward $\pi_* \mathcal{O}_Z$ along $\pi : Z \to S$ carries a canonical section $s_{ij} : \mathcal{O}_S \to \pi_* \mathcal{O}_Z$ for each monomial $x^i y^j$. We can take the direct sum

$$s_\lambda = \sum_{(i,j)\in\lambda} s_{ij} : \mathcal{O}_S^{\oplus n} \to \pi_* \mathcal{O}_Z.$$

Then $U_\lambda$ is the subfunctor representing those $S$-points such that $s_\lambda$ is an isomorphism. The points of $U_\lambda$ are exactly those ideals $I$ such that $B_\lambda$ is a basis for $\mathbb{C}[x,y]/I$.

**Proposition 1.** *$U_\lambda$ is a $T$-invariant open affine neighborhood of $I_\lambda$.*

*Proof.* It is clear that $[I_\lambda] \in U_\lambda$ and that $U_\lambda$ is $T$-invariant. We will write down explicit coordinates for $U_\lambda$. Given a monomial $x^r y^s$ we have a unique expansion

$$x^r y^s = \sum_{(i,j)\in\lambda} c_{ij}^{rs}(I) x^i y^j \mod I$$

for any $I \in U_\lambda$ where $c_{ij}^{rs}(I)$ are coefficients depending on $I$. The $c_{ij}^{rs}$ are in fact global sections of $\mathcal{O}_{U_\lambda}$. To see this, using the notation above, note that for any $S$-point $(Z \subset S \times \mathbb{A}^2)$ of $U_\lambda$ and any monomial $x^r y^s$, we have a section $s_{rs} : \mathcal{O}_S \to \pi_* \mathcal{O}_Z$. Pulling back by the isomorphism $s_\lambda$, we obtain a section $s_\lambda^* s_{rs} : \mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ where the components of the target are indexed by $(i,j) \in \lambda$. Then the functions $c_{ij}^{rs}$ on $S$ are exactly the components of $s_\lambda^* s_{rs}$.

Since $I$ is an ideal, it is closed under multiplication by $x$ and $y$. Multiplying the above equation $x$ and $y$ respectively, re-expanding both sides in the basis $B_\lambda$, and equating coefficients gives us the following.

$$c_{ij}^{r+1,s} = \sum_{(h,k)\in\lambda} c_{hk}^{rs} c_{i,j}^{h+1,k} \quad (1)$$

$$c_{ij}^{r,s+1} = \sum_{(h,k)\in\lambda} c_{hk}^{rs} c_{ij}^{h,k+1} \quad (2)$$

Now we leave it to the reader to check that $U_\lambda$ is represented by Spec of the ring

$$\mathcal{O}_\lambda := \mathbb{C}[c_{ij}^{rs} \mid (i,j) \in \lambda]/(\text{relations (1) \& (2)}).$$

□

**Remark 1.** *In fact the $U_\lambda$ are the pullbacks of the natural open affine subfunctors that cover a Grassmannian under the embedding of the Hilbert scheme into a Grassmannian used to construct $\mathbf{H}^n$. In particular, $U_\lambda$ over all $\lambda$ cover $\mathbf{H}^n$.*

Now we will compute the cotangent space to $\mathbf{H}^n$ at a monomial ideal $I_\lambda$. For this ideal, we have

$$c_{ij}^{rs}(I_\lambda) = \begin{cases} 1 & (i,j) = (r,s) \in \lambda \\ 0 & \text{else} \end{cases} \blacksquare$$

Thus the maximal ideal $\mathfrak{m}_\lambda \subset \mathcal{O}_\lambda$ corresponding to the point $[I_\lambda] \in U_\lambda \subset \mathbf{H}^n$ is generated by $c_{ij}^{rs}$ for $(r,s) \notin \lambda$. The cotangent space to an affine scheme is given by $\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2$. Examining the relations above, we see that all the terms on the right are in $\mathfrak{m}_\lambda^2$ except for the term

$$c_{i-1,j}^{rs} c_{ij}^{ij} = c_{i-1,j}^{rs}.$$

Here we are using that $c_{ij}^{ij} = 1$. Thus we have that

$$c_{ij}^{r+1,s} = c_{i-1,j}^{rs} \mod \mathfrak{m}_\lambda^2.$$

Similarly, $c_{ij}^{r,s+1} = c_{i,j-1}^{rs} \mod \mathfrak{m}_\lambda^2$. For each box $(i,j) \in \lambda$ we define two special functions $u_{ij}$ and $d_{ij}$ as in the following diagram.

put in picture and discussion of arrows

Now a simple combinatorial argument shows that each function is either zero or equivalent to one of the $d_{ij}$ or $u_{i,j}$ in $\mathfrak{m}_\lambda^2$. Since there are $2n$ such functions, we conclude the following.

**Proposition 2.** *The cotangent space to $\mathbf{H}^n$ at $I_\lambda$ has dimension at most $2n$.*

## 2.3 Initial degenerations

Let $\rho : \mathbb{G}_m \to T$ be a character of the torus so that $\rho(t) = (t^a, t^b)$. Then we define the initial ideal, if it exists, to be the flat limit

$$in_\rho I := \lim_{t\to 0} \rho(t) \cdot I.$$

More precisely, the action of $T$ on $\mathbf{H}^n$ composed with $\rho$ induces an action of $\mathbb{G}_m$. Then the orbit of $I$ can be viewed as a morphism

$$\varphi_I : \mathbb{G}_m \to \mathbf{H}^n$$

corresponding to the family of ideals $I_\rho = (\{f(t^a x, t^b y) \mid f \in I\}) \subset \mathcal{O}_{\mathbb{G}_m}[x,y]$. If this morphism extends to an equivariant morphism

$$\bar{\varphi}_I : \mathbb{A}^1 \to \mathbf{H}^n,$$

the initial ideal is exactly the ideal corresponding to the point $\bar{\varphi}_I(0)$.

**Fact 1.** *There exists a generic enough $\rho$ such that the fixed points of the $\mathbb{G}_m$ action under $\rho$ are the same as those for $T$, $(\mathbf{H}^n)^\rho = (\mathbf{H}^n)^T$, and such that $\bar{\varphi}_I$ exists for each $I$.*

**Exercise 1.** *Check that the co-character $\rho(t) = (t^{-p}, t^{-q})$ for $p \gg q > 0$ works.*

Since $\bar{\varphi}_I$ has $\mathbb{G}_m$-equivariant it sends fixed points to fixed points so $\bar{\varphi}_I(0) \in (\mathbf{H}^n)^T$ is a monomial ideal.

**Proposition 3.** *$\mathbf{H}^n$ is connected.*

*Proof.* By the above fact, every point of $\mathbf{H}^n$ is connected to a monomial ideal by an initial degeneration over $\mathbb{A}^1$. Thus, it suffices to show that the monomial ideals lie in the same connected component. Let $\lambda_i$ be partitions corresponding to ideals $I_i$ for $i = 1, 2$. Suppose the partitions differ in exactly one box.

$$\lambda_2 = (\lambda_1 \setminus (i, j)) \cup (r, s)$$

Let $J = I_1 \cap I_2$ corresponding to the partition $\mu = \lambda_1 \cup \lambda_2 \subset \mathbb{N}^2$. Then the family of ideals

$$I_{\alpha,\beta} = J + (\alpha x^i y^j - \beta x^r y^s)$$

gives map $\varphi : \mathbb{P}^1 \to \mathbf{H}^n$ with $\varphi(0, 1) = I_1$ and $\varphi(1, 0) = I_2$. Now any partition can be obtained from the row partition $(n)$ by moving one box at a time. This shows each monomial ideal is in the same connected component as the row so $\mathbf{H}^n$ is connected.

□

**Example 1.** *Let $\lambda_1 = (1, 1, 1, 1)$ and $\lambda_2 = (2, 1, 1)$ so that the ideals $\lambda_i$ differ by one box. These correspond to the ideals $I_1 = (x, y^4)$ and $I_2 = (x^2, xy, y^3)$ respectively. Consider the ideal $J = I_1 \cap I_2 = (x^2, xy, y^4)$ corresponding to the partition $\mu = \lambda_1 \cup \lambda_2 \subset \mathbb{N}^2$. Then the one parameter family $(x^2, xy, y^4, \alpha y^3 - \alpha x)$ of ideals connects these two points of $\boldsymbol{H}^4$.*

**Remark 2.** *Note that the curves in $\boldsymbol{H}^n$ constructed above to connect two monomial ideals parametrize subschemes $Z$ supported on the origin. That is, such that $Z_{red} = \{(0, 0)\}$.*

## 2.4 The Hilbert-Chow morphism for $\mathbb{A}^2$