# Lecture 23-24: The moduli of curves

## 1 The functor of genus $g$ curves

First try: ($g \geq 2$ morphisms are automatically projective)

**Definition 1.** *A smooth curve over $S$ is a flat and proper morphism $f : X \to S$ with smooth geometrically connected 1-dimensional fibers. The genus of $X \to S$ is the genus of a geometric fiber.*[1]

$$\begin{aligned}\pi_0\mathcal{M}_g &: Sch_{\mathbb{Z}} \to Set \\ S &\to \{f : X \to S \text{ a smooth curve of genus } g\}/\sim\end{aligned}$$

$\pi_0\mathcal{M}_g$ is *not representable.*

**Example 1.** $C \times \mathbb{P}^1 \to$ *node.*

Fix, upgrade the functor to a *pseudofunctor*

$$\mathcal{M}_g : Sch_S \to Gpd$$

Define groupoids + Stacks = pseudofunctors to groupoid + sheaf.
Diagram relating all notions. Explain notation $\pi_0\mathcal{M}_g$.

## 2 Stacks

**Definition 2.** *Category fibered in groupoids (CFG) $p : \mathscr{X} \to \mathcal{C}$ such that blah. If $f : T' \to T$ in , and $E$ an object over $T$, then there exists a $E'$ unique up to unique isom and a map $E' \to E$ lying over $f$.*

Denote $E' = f^*E$. $p^{-1}(T) :=$ objects over $T$ + morphisms over $id_T$. Makes precise the idea of a "pseudofunctor" to groupoids. $T \mapsto p^{-1}(T)$ which is a groupoid. We will denote $p^{-1}(T)$ by $\mathscr{X}_T$. Presheaves are CFG by viewing a set as a category with only identities. Objects $S$ may be indentified with the category $\mathcal{C}/S$ (equivalent to the data of the functor of points of $S$) and maps $S \to \mathscr{X}$ identified with objects of $\mathscr{X}_S$ by where $id : S \to S$ maps. There is a 2-categorical Yoneda lemma.

**Example 2.** *$B\mathbb{G}_m$, $BGL_n$, quotient stack, Picard stack, $\mathcal{M}_g$ as a CFG.*

[1]Note this is constant over connected components of $S$ by flatness.

**Fact 1.** *Fiber products of CFGs exist. I'll let you work out the details of the definition.*

Consider $Sch_S$ with a Grothendieck topology $\mathcal{T} =$ (Zariski, étale, fppf, fpqc, etc).

**Definition 3.** *A $\mathcal{T}$-stack is a category $p : \mathscr{X} \to Sch_S$ over $Sch_S$ such that*

*(1) $p$ is a CFG,*

*(2) for each scheme $T \to S$ and each pair of objects $\xi, \psi \in \mathscr{X}_T$, the functor $Sch_T \to Set$ given by $f : V \to T$ maps to*

$$\mathrm{Hom}_{\mathscr{X}_V}(f^*\xi, f^*\psi)$$

*is a $\mathcal{T}$-sheaf, and*

*(3) objects of $\mathcal{X}$ satisfy effective $\mathcal{T}$-descent.*

**Example 3.** *All examples above are fppf stacks (and thus also Zariski and étale) (need $g \neq 1$).*

A morphism of stacks is representable by schemes if the usual thing. Can define all properties $\mathcal{P}$ for representable morphisms.[2]

# 3 Algebraic stacks

From now on work with étale or fppf topology, won't make a difference which.

**Lemma 1.** *Let $\mathscr{X}$ be a stack over $Sch_S$. Then the diagonal map*

$$\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

*is representable by schemes if and only if for all schemes $T_1, T_2 \to \mathscr{X}$, the fiber product $T_1 \times_{\mathscr{X}} T_2$ is a scheme.*

That is, the diagonal is representable by schemes if and only if for any morphism $T \to \mathscr{X}$ from a scheme is representable. For a stack $\mathscr{X}$ with representable diagonal, we can define all the usual separation axioms.

**Remark 1.** *How do we check if $\Delta$ is representable? We need to show that for any $T \to \mathscr{X} \times_S \mathscr{X}$, the pullback $T \times_{\mathscr{X} \times_S \mathscr{X}, \Delta} \mathscr{X}$ is a scheme. $T \to \mathscr{X} \times_S \mathscr{X}$ corresponds to a pair of objects $\xi, \psi \in \mathscr{X}(T)$ over $T$ as well as an isomorphism $\xi \to \psi$, that is, an element of $\mathcal{H}om_{\mathscr{X}(T)}(\xi, \psi) = Isom_T(\xi, \psi)$. By definition of a stack, the functor sending a $T' \to T$ to $\mathrm{Isom}_{T'}(\xi_{T'}, \psi_{T'})$ is a sheaf which is isomorphic to the pullback*

$$T \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}.$$

*Thus the condition that the diagonal is representable is the condition that for any $T$ and any objects $\xi, \psi$ over $T$, the isom sheaf is representable by a scheme.*

**Definition 4.** *A stack $\mathscr{X}$ is an algebraic stack (resp. Deligne-Mumford stack) if*

*(1) the diagonal $\Delta_{\mathscr{X}}$ is representable$^{**}$, and*

*(2) there exists a scheme $U$ and a smooth (resp. étale) surjection $U \to \mathscr{X}$.*

[2]something about representable = representable by spaces

**Remark 2.** *(A remark on algebraic spaces) If $\mathscr{X}$ is a stack where the groupoids are sets, that is, $\mathscr{X}$ is simply a sheaf, that satisfies the conditions of the above theorem, then we say that the sheaf $\mathscr{X}$ is an algebraic space. In this case, $\mathscr{X}$ is the quotient in the category of sheaves of the equivalence relation*

$$U \times_{\mathscr{X}} U \rightrightarrows U$$

*of schemes. Once one develops the theory of algebraic spaces, then the right notion of a representable map of stacks is one that is representable by algebraic spaces, rather than representable by schemes.*

**Remark 3.** *Let us unravel the definition, we need $\mathscr{X}$ to be a sheaf so that we can do geometry locally, we need representability of the diagonal to make sense of the having a smooth or étale cover by a scheme, and then we can use descent by this cover to "do geometry" on $\mathscr{X}$.*

Algebraic stacks have a Zariski topology generated by morphisms that are representable by open immersions, and an underlying topological space $|\mathscr{X}|$ given by equivalence classes of $K$-points for fields $K$ and the Zariski topology. Universally closed makes sense with the usual definition that for any $\mathscr{Z}$, the map $|\mathscr{X} \times_S \mathscr{Z}| \to |\mathscr{Y} \times_S \mathscr{Z}|$ is closed.

**Theorem 1.** *(Valuative criterion for properness) Proper = universally closed + separated by defition. Diagram... Existence + uniqueness separate.*

**Theorem 2.** *Suppose $\mathscr{X}$ is a quasi-separated algebraic stack such that $\Delta_{\mathscr{X}}$ is unramified. Then $\mathscr{X}$ is a Deligne-Mumford stack.*

Definition of coarse moduli space... unique up to unique iso. Example...

**Theorem 3.** *(Keel-Mori) Suppose $\mathscr{X}$ is a proper Deligne-Mumford stack over $S$. Then there exists a proper coarse moduli space $X \to S$.*

The CFG of stable curves $\overline{\mathcal{M}}_g$

**Theorem 4.** *(Deligne-Mumford) $\overline{\mathcal{M}}_g$ for $g \geq 2$ is a smooth and proper Deligne-Mumford stack of dimension $3g - 3$ with projective coarse moduli space $\overline{M}_g$.*