Lecture 3: Grassmannians (cont.) and flat morphisms

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1 Constructing Grassmannians

Recall we aim to prove the following:

Theorem 1. *$Gr(k,n)$ is representable by a finite type scheme over* Spec $\mathbb{Z}$.

Proof. We will use the representability criterion from Lecture 2. It is clear $Gr(k,n)$ is a sheaf by gluing locally free sheaves.

For each subset $i \subset \{1,\ldots,n\}$ of size k, we will define a subfunctor F_i of the Grassmannian functor. First, let

$$s_i : \mathcal{O}_S^k \to \mathcal{O}_S^n$$

denote the the inclusion where the j^{th} direct summand is mapped by the identity to the i_j^{th} direct summand. Now let F_i be defined as the subfunctor

$$F_i(S) = \{\alpha : \mathcal{O}_S^n \to \mathcal{V} \mid \alpha \circ s_i \text{ is surjective } \} \subset Gr(k,n)(S).$$

Note this is a functor since for any $f : T \to S$, f^* is right-exact.

We need to show that each F_i is representable and that the collection $\{F_i\}$ is an open cover of the functor $Gr(k,n)$.

For any scheme S and any map $S \to Gr(k,n)$ corresponding to the object $(\alpha : \mathcal{O}_S^n \to V) \in F(S)$, we have a natural morphism of finite type quasi-coherent sheaves $\alpha \circ s_i : \mathcal{O}_S^n \to \mathcal{V}$. Now let $\mathcal{K}$ be the cokernel of $\alpha \circ s_i$. Then $\alpha \circ s_i$ is surjective at a point $x \in S$ if and only if $\mathcal{K}_x = 0$ if and only if $x \notin \mathrm{Supp}(\mathcal{K}_x)$. Since $\mathrm{Supp}(\mathcal{K}_x)$ is closed, the set U_i where $\alpha \circ s_i$ is surjective is open.

We need to show that for any other scheme T and a morphism $f : T \to S$, f factors through U_i if and only if

$$f^*(\alpha : \mathcal{O}_S^n \to \mathcal{V}) = (f^*\alpha : \mathcal{O}_T^n \to f^*\mathcal{V}) \in F_i(T)$$

. Suppose $t \in T$ maps to $x \in S$. By Nakayama's lemma, $x \in U_i$ if and only if

$$(\alpha \circ s_i)|_x : k(x)^k \to \mathcal{V}_x/\mathfrak{m}_x\mathcal{V}_x$$

is surjective. Now the stalk $(f^*\mathcal{V})_t$ is given by the pullback $\mathcal{V}_x \otimes_{\mathcal{O}_{S,x}} \mathcal{O}_{T,t}$ along the local ring homomorphism $f^* : \mathcal{O}_{S,x} \to \mathcal{O}_{T,t}$. Thus the map on fibers

$$(f^*\alpha \circ f^*s_i)|_t : k(t)^k \to (f^*\mathcal{V}_t)/\mathfrak{m}_t\mathcal{V}_t$$

is the pullback of $(\alpha \circ s_i)|_x$ by the residue field extension $k(x) \subset k(t)$. In particular, one is surjective if and only if the other is so $f : T \to S$ factors through U_i if and only if $(f^*\alpha \circ f^*s_i)|_t$ is surjective for all $t \in T$ if and only if $f^*(\alpha : \mathcal{O}_S^k \to \mathcal{V}) \in F_i(T)$. This proves that the F_i are open subfunctors.

Next, we need to know that the collection $\{F_i\}$ covers F. This amounts to showing that for any $S \to Gr(k,n)$ as above and any $s \in S$, there exists an i such that $s \in U_i$. As above, by Nakayama's lemma we may check surjectivity on fibers, thus we need to show that for each $s \in S$, there exists an i such that the composition

$$k(s)^k \xrightarrow{s_i} k(s)^n \xrightarrow{\alpha} \mathcal{V}_s / \mathfrak{m}_s \mathcal{V}_s$$

but this is clear from linear algebra.

Finally, we need to show that the F_i are representable. Given $(\alpha : \mathcal{O}_S^n \to \mathcal{V}) \in F_i(S)$, the composition $\alpha \circ s_i : \mathcal{O}_S^k \to \mathcal{V}$ is a surjection between finite locally free modules of the same rank. Then we apply the following lemma from commutative algebra.

Lemma 1. *Let R be a ring and M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

By applying the lemma to an open cover of S where $\mathcal{V}$ is trivial, we get that $\alpha \circ s_i$ is actually an isomorphism. In particular, we obtain a splitting of α so that α is determined by its restriction to the complimentary $n-k$ components of $\mathcal{O}_S^n$. The restriction of α to each component is a map $\mathcal{O}_S \to \mathcal{O}_S^k$ which is the same as a k-tuple of sections. Thus, the functor F_i is isomorphic to the functor of a $(n-k)$ many k-tuples of sections, i.e. to $\mathbb{A}_{\mathbb{Z}}^{k(n-k)}$.

□

1.1 Some properties of the Grassmannian

Next we want to use the functorial point of view to show that $Gr(k,n)$ is in fact a projective variety. First we review the valuative criterion of properness.

Theorem 2. *Suppose $f : X \to Y$ is a locally of finite type morphism of schemes with Y locally Noetherian. Then the following are equivalent:*

(a) f is separated (resp. universally closed, resp. proper),

(b) for any solid commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

where R is a DVR with fraction field K, any dashed arrow is unique (resp. there exists a dashed arrow, resp. there exists a unique dashed arrow).

The nice thing about this is that condition (b) can be phrased completely in terms of the functor of points: for any DVR Spec $R \to Y$ over Y with fraction field K, the map of sets

$$\operatorname{Hom}_Y(\operatorname{Spec} R, X) \to \operatorname{Hom}_Y(\operatorname{Spec} K, X)$$

is injective (resp. surjective, resp. bijective). This is useful to prove properness of moduli spaces directly from the moduli functor. It amounts to saying that any family over Spec K can be uniquely "filled in" over the the closed of point of Spec R to a family over Spec R

Proposition 1. *$Gr(k,n)$ is proper over* Spec $\mathbb{Z}$*.*

Proof. By construction, $Gr(k,n)$ has a finite cover by finite dimensional affine spaces over Spec $\mathbb{Z}$ so it is of finite type of over Spec $\mathbb{Z}$. By the valuative criterion we need to check that we can uniquely fill in the dashed arrow in the following diagram.

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & Gr(k,n) \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } R & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

The top morphism is a surjection of K-vector spaces

$$K^n \to V$$

where V has rank k. We have a natural inclusion of R-modules $R^n \to K^n$. We need to find a locally free rank k module M over R with a surjection $R^n \to M$ such that the following diagram commutes.

$$\begin{array}{ccc} K^n & \longrightarrow & V \\ \uparrow & & \uparrow \\ R^n & \longrightarrow & M \end{array}$$

Then we can just take M to be the image of $R^n \subset K^n \to V$. Since V is a vector space over K, it is torsion free as an R-module so the finitely generated M is free. By construction $M \otimes_R K = V$ so M has rank k and M is clearly unique. $\square$

Next we will show that $Gr(k,n)$ is in fact projective over Spec $\mathbb{Z}$!. To do this we will use the following criterion:

Proposition 2. *Suppose $f : X \to Y$ is a proper monomorphism of schemes. Then f is a closed embedding.*

Proposition 3. *$Gr(k,n)$ is smooth and projective over* Spec $\mathbb{Z}$*.*

Proof. We will show that $Gr(k,n)$ is a subfunctor of projective space. Then combining the previous two propositions, it follows that $Gr(k,n)$ is a closed subscheme of projective space. For smoothness, we saw in the construction that $Gr(k,n)$ is covered by open subsets isomorphic to $\mathbb{A}^{k(n-k)}$, so it is smooth.

We define a natural transformation of functors $Gr(k,n) \to \mathbb{P}^N_{\mathbb{Z}}$ where $N = \binom{n}{k} - 1$. For any scheme S and any S-point $(\alpha : \mathcal{O}_S^n \twoheadrightarrow \mathcal{V}) \in Gr(k,n)(S)$, we consider the induced map on the k^{th} alternating power.

$$\Lambda^k \alpha : \Lambda^k \mathcal{O}_S^n = \mathcal{O}_S^{\binom{n}{k}} \to \Lambda^k \mathcal{V}$$

Since α is surjective, so is $\Lambda^k \alpha$. Moreover, $\Lambda^k \mathcal{V}$ is a line bundle since the rank $\mathcal{V}$ is k. Finally, Λ^k commutes with pullbacks. Thus $(\Lambda^k \alpha : \mathcal{O}_S^{\binom{n}{k}} \to \Lambda^k \mathcal{V}) \in \mathbb{P}^N_{\mathbb{Z}}(S)$ is an S point of $\mathbb{P}^N_{\mathbb{Z}}$.

We need to show that the natural transformation $Gr(k,n) \to \mathbb{P}^N_{\mathbb{Z}}$ is a subfunctor. We will do this by restricting to the open subfunctors $F_i \subset Gr(k,n)$ described in the proof that $Gr(k,n)$ is representable. Let G_i be the corresponding open subfunctors of $\mathbb{P}^N$ (i.e. the subfunctor where the i^{th} section is nonvanishing, or equivalently where the composition $\mathcal{O}_S \to \mathcal{O}_S^{N+1} \to L$ with the i^{th} copy is surjective).

It is clear that the natural transformation above maps F_i to G_i. Thus it suffices to show that F_i is a subfunctor of G_i. Now F_i is the functor corresponding to $k \times n$ matrices of global sections of $\mathcal{O}_S$ where the columns indexed by the subset i form the identity matrix, and the natural transforation $F_i \to G_i$ is given by taking $k \times k$ minors of this matrix. Now it is an exercise in linear algebra to see that the minors of such a matrix uniquely determine the matrix. Thus $F_i \to G_i$ is a subfunctor so $Gr(k,n) \to \mathbb{P}^N$ is a monomorphism. □

Over the Grassmannian we have the universal quotient

$$\mathcal{O}^n_{Gr(k,n)} \to \mathcal{Q}.$$

The above proof shows in fact that $\Lambda^k \mathcal{Q} =: \det \mathcal{Q}$ is a very ample line bundle on $Gr(k,n)$ which induces the closed embedding into projective space.

1.2 Relative Grassmannians

For any scheme S, we can basechange from Spec $\mathbb{Z}$ to S to get the scheme representing the Grassmannian functor $Sch_S \to Set$. More generally, if $\mathcal{E}$ is any rank n vector bundle on a scheme S, we can define for any $k < n$, the Grassmannian of $\mathcal{E}$ as the functor $Sch_S \to Set$ given by

$$Gr_S(k,\mathcal{E})(f : T \to S) = \{\alpha : f^*\mathcal{E} \twoheadrightarrow \mathcal{V} \mid \mathcal{V} \text{ is a rank } k \text{ locally free sheaf}\}$$

Theorem 3. *$Gr_S(k,\mathcal{E})$ is representable by a smooth projective scheme over S.*

Proof. We leave the details to the reader, but the idea is to cover S by open subsets where $\mathcal{E}$ is locally free. Over these subsets the Grassmannian functor is representable by the argument above. Then we may glue these schemes together. The properties of being proper and being a monomorphism are both local on the base and compatible with base change. Putting this all together we get a scheme representing $Gr_S(k,\mathcal{E})$ as well as a closed embedding into $\mathbb{P}_S(\Lambda^k \mathcal{E})$. □

Remark 1. *(A note on projectivity) Let $f : X \to S$ be a morphism of schemes. There are several notions of projectivity for f.*

(a) there is a closed embedding of X into $\mathrm{Proj}_S \mathrm{Sym}^*_{\mathcal{O}_S} \mathcal{F}$ for some coherent sheaf $\mathcal{F}$ on S;

(b) there is a closed embedding of X into $\mathbb{P}_S(\mathcal{E})$ for some finite rank locally free sheaf $\mathcal{E}$ on S;

(c) there is a closed embedding of X into $\mathbb{P}^n_S$.

For each notion of projectivity we can define quasi-projective morphisms as those which factor through an open embedding into a projective one. The above theorem shows that relative Grassmannians $Gr_S(k,\mathcal{E})$ are projective over S in the sense of (b). We have implications $(c) \implies (b) \implies (a)$ but *these notions are not equivalent in general. If S is Noetherian and satisfies the resolution property*[1] *then* $(a) \implies (b)$ *and if furthermore S admits an ample line bundle, then* $(b) \implies (c)$. *For simplicity we will usually use projective to mean* (c).

2 Recollections on flaness

We want to move on to moduli spaces of general varieties. To do this we need a good notion of continuously varying family of varieties (or schemes, or sheaves) parametrized by a base scheme S. It turns out the right notion is that of *flatness*.

Definition 1. *Let $f : X \to S$ be a morphism of schemes and $\mathcal{F}$ a quasi-coherent sheaf on X. We say that $\mathcal{F}$ is flat over S at $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module. We say that $\mathcal{F}$ is flat over S if it is flat at every point $x \in X$. We say that the morphism f is flat if $\mathcal{O}_X$ is flat over S.*

We recall some basic facts about flatness.

Proposition 4.

1. *The property of being flat is stable under base change and compositions;*
2. *localizations of flat modules are flat so in particular open embeddings are flat;*
3. *if R is a PID, then an R-module M is flat if and only if it is torsion free;*
4. *if $t \in \mathcal{O}_{S,f(x)}$ is a non-zero divisor, then $f^*t \in \mathcal{O}_{X,x}$ is a non-zero divisor;*
5. *if S is Noetherian and f is finite then $\mathcal{F}$ is flat if and only if $f_*\mathcal{F}$ is locally free of finite rank.*

Recall that an associated point of a scheme X is a point $x \in X$ so that the corresponding prime ideal $\mathfrak{m}_x$ is generated by zero divisors. If X is reduced, then these are just the generic points of irreducible components of X.

Proposition 5. *Let $f : X \to C$ be a morphism of schemes with C an integral regular scheme of dimension 1. Then f is flat if and only if it maps all associated points of X to the generic point of C.*

References

[1] That is, if every coherent sheaf admits a surjection from a finite rank locally free sheaf.