Lecture 4: Flat morphisms and Hilbert polynomials

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1 More on flat morphisms

Last time we left off with the following statement.

Proposition 1. *Let $f : X \to Y$ be a morphism of schemes with Y an integral regular scheme of dimension 1. Then f is flat if and only if it maps all associated points of X to the generic point of Y.*

Proof. Suppose f is flat and take $x \in Y$ with $f(x) = y$ a closed point. Then $\mathcal{O}_{Y,y}$ is a DVR with uniformizing parameter $t_y \in \mathfrak{m}_y$. Since t_y is a non-zero divisor, $f^* t_y \in \mathfrak{m}_x$ is a non-zero divisor so x is not associated.

Conversely, if f is not flat, there is some $x \in X$ with $y = f(x)$ a closed point and $\mathcal{O}_{X,x}$ is not a flat $\mathcal{O}_{Y,y}$ module. Since $\mathcal{O}_{Y,y}$ is a DVR, this means $\mathcal{O}_{X,x}$ is not torsion free so $f^* t_y$ is a zero divisor which must be contained in some associated prime mapping to y.

□

Corollary 1. *Let $X \to Y$ as above. Then f is flat if and only if for each $y \in Y$, the scheme theoretic closure of $X \setminus X_y$ inside X is equal to X.*

The slogan to take away from the above corollary is that flat morphisms over a smooth curve are continuous in the following sense:

$$\lim_{y \to y_0} X_y = X_{y_0}$$

for each point $y_0 \in C$.

Corollary 2. *Let Y be as above and $y \in Y$. Suppose $X \subset \mathbb{P}^n_{Y \setminus y}$ is flat. Then there exists a unique subscheme $\overline{X} \subset \mathbb{P}^n_Y$ such that $\overline{X} \to Y$ is flat.*

In particular, the functor of flat subschemes of a projective scheme satisfies the valuative criterion of properness!

Example 1. *Consider the subscheme $X \subset \mathbb{P}^3_{\mathbb{A}^1_a \setminus 0}$ defined by the ideal*

$$I = (a^2(xw + w^2) - z^2, ax(x + w) - yzw, xz - ayw).$$

For each $a \neq 0$, this is the ideal of the twisted cubic which is the image of the morphism

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [s, t] &\mapsto [t^2 s - s^3, t^3 - ts^2, ats^2, s^3]. \end{aligned}$$

By the above Corollary, we can compute the flat limit

$$\lim_{a\to 0} X_a = \overline{X}_0$$

by computing the closure $\overline{X}$ *of* X *in* $\mathbb{P}^3_{\mathbb{A}^1}$. *We can do this by taking* $a \to 0$ *in the ideal* I *but we have to be careful! Note that the polynomial*

$$y^2w - x^2(x+w)$$

is contained in the ideal I. *In fact*

$$I/aI = (z^2, yz, xz, y^2w - x^2(x+w))$$

which gives the flat limit of this family of twisted cubics. Note that set theoretically this is a nodal cubic curve in the $z = 0$ *plane but at* $[0,0,0,1]$ *it has an embedded point that "sticks out" of the plane.*

The following is an interesting characterization of flatness over a reduced base.

Theorem 1 (somewhere in ega)**.** *(Valuative criterion for flatness) Let* $f : X \to S$ *be a locally of finite presentation morphism over a reduced Noetherian scheme* S. *Then* f *is flat at* $x \in X$ *if and only if for each DVR* R *and morphism* $\operatorname{Spec} R \to S$ *sending the closed point of* $\operatorname{Spec} R$ *to* $f(s)$, *the pullback of* f *to* $\operatorname{Spec} R$ *is flat at all points lying over* x.

We will see a proof of this in the projective case soon.

Proposition 2. *Let* $f : X \to Y$ *be a flat morphism of finite type and suppose* Y *is locally Noetherian and locally finite-dimensional. Then for each* $x \in X$ *an* $y = f(x)$,

$$\dim_x(X_y) = \dim_x(X) - \dim_y(Y).$$

Proof. It suffices to check after base change to $\operatorname{Spec} \mathcal{O}_{Y,y}$ so suppose Y is the spectrum of a finite dimensional local ring. We will induct on the dimension of Y. If $\dim(Y) = 0$, then $X_y = X_{red}$ so there is nothing to check. If $\dim(Y) > 0$, then there is some non-zero divisor $t \in \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ so that $f^*t \in \mathfrak{m}_x$ is a non-zero divisor. Then the induced map $X' = \operatorname{Spec} \mathcal{O}_{X,x}/f^*t \to Y' = \operatorname{Spec} \mathcal{O}_{Y,y}/t$ is flat, $\dim(X') = \dim_x(X) - 1$, $\dim(Y') = \dim(Y) - 1$, and the result follows by induction. □

Corollary 3. *If* X *and* Y *are integral* k*-schemes, then* $n = \dim(X_y)$ *is constant for* $y \in \operatorname{im}(f)$ *and* $\dim(X) = n + \dim(Y)$.

2 Hilbert polynomials

Let $X \subset \mathrm{P}^n_k$ be a projective variety over a field k. Recall that the *Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ on X may be defined as

$$P_{\mathcal{F}}(d) := \chi(X, \mathcal{F}(d)) := \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{F}(d))^1$$

[1] It is not a priori clear that this is a polynomial n. To prove this, one can induct on the dimension of X and use the additivity of Euler characteristics under short exact sequences.

where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes d}$. By the Serre vanishing theorem,

$$\chi(X, \mathcal{F}(d)) = \dim H^0(X, \mathcal{F}(d))$$

for $n \gg 0$. When $\mathcal{F} = \mathcal{O}_X$, then we call $P_X(d) := P_{\mathcal{O}_X}(d)$ the Hilbert polynomial of X.

We have the following important theorem.

Theorem 2. *Let $f : X \to Y$ be a projective morphism over a locally Noetherian scheme Y. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $P_{\mathcal{F}|_{X_y}}(d)$ is locally constant for $y \in Y$. If Y is reduced, then the converse holds.*

Proof. By pulling back along the inclusion Spec $\mathcal{O}_{Y,y} \to Y$, we may assume that $Y = \text{Spec } A$ is the spectrum of a Noetherian local ring. Moreover, by considering the pushforward $i_*\mathcal{F}$ under the map $i : X \hookrightarrow \mathbb{P}^n_Y$, we may assume that $X = \mathbb{P}^n_Y$. We have the following lemma:

Lemma 1. *$\mathcal{F}$ is flat over Y if and only if $H^0(X, \mathcal{F}(d))$ is a finite free A-module for $d \gg 0$.*

Proof. $\implies$: Let $\mathcal{U} = \{U_i\}$ be an affine open covering of X and consider the Čech complex

$$0 \to H^0(X, \mathcal{F}(d)) \to C^0(\mathcal{U}, \mathcal{F}(d)) \to C^1(\mathcal{U}, \mathcal{F}(d)) \to \ldots \to C^n(\mathcal{U}, \mathcal{F}(d)) \to 0.$$

By Serre vanishing, this sequence is exact for $d \gg 0$. Since $\mathcal{F}$ is flat, each term $C^i(\mathcal{U}, \mathcal{F}(d))$ is a flat finitely generated A-module. We repeatedly apply the following fact: if $0 \to A \to B \to C \to 0$ is exact and B and C are flat, then A is flat. It follows that $H^0(X, \mathcal{F}(d))$ is a finitely generated flat module over the local ring A, and in particular, is free.

$\impliedby$: Suppose that d_0 is such that $H^0(X, \mathcal{F}(d))$ is finite and free for $d \geq d_0$ and consider the $S = A[x_0, \ldots, x_n]$ module

$$M = \bigoplus_{d \geq d_0} H^0(X, \mathcal{F}(d)).$$

Now M is A-flat since it's a direct sum of flat modules. Furthermore, M defines a quasicoherent sheaf $\tilde{M}$ on X which is just $\mathcal{F}$ itself. Explicitly, $\tilde{M}$ is obtained by gluing together the degree 0 parts of the localizations of M by each x_i. Since flatness is preserved by localization and direct summands of flat modules are flat, we conclude that $\tilde{M} = \mathcal{F}$ is flat. □

Now the first part of the theorem would follow if we know that the rank of the A-module $H^0(X, \mathcal{F}(d))$ equals $P_{\mathcal{F}|_{X_y}}(d)$. This is implied by the following equality base change statement.

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) = H^0(X_y, \mathcal{F}(d)|_{X_y}) \tag{1}$$

Remark 1. *One can rewrite equality 1 as saying the natural map*

$$u^* f_*(\mathcal{F}(d)) \to f'_* u'^*(\mathcal{F}(d))$$

where

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{2}$$

is the Cartesian diagram with $i : Y' = \text{Spec } k(y) \to Y$ the inclusion. More generally, given any Cartesian diagram as above and any quasicoherent sheaf $\mathcal{F}$ on X, there are natural maps

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

One can ask more generally if this map is an isomorphism, and if it is we say that base change holds (for this diagram, this sheaf, and this i), or that the i^{th} cohomology of $\mathcal{F}$ commutes with base change by u. We highlight this here since this situation will come up again.

Suppose first that $y \in Y$ is a closed point. Then consider a resolution of $k(y)$ of the form

$$A^m \to A \to k(y) \to 0. \tag{3}$$

Pulling back and tensoring with $\mathcal{F}$ we get a resolution

$$\mathcal{F}^m \to \mathcal{F} \to \mathcal{F}|_{X_y} \to 0.$$

For $d \gg 0$ and by Serre vanishing, the sequence

$$H^0(X, \mathcal{F}(d)^{\oplus m}) \to H^0(X, \mathcal{F}(d)) \to H^0(X_y, \mathcal{F}(d)|_{X_y}) \to 0$$

is exact. On the other hand, we can tensor sequence 3 by the A-module $H^0(X, \mathcal{F}(d))$ to get an exact sequence

$$H^0(X, \mathcal{F}(d))^{\oplus m} \to H^0(X, \mathcal{F}(d)) \to H^0(X, \mathcal{F}(d)) \otimes_A k(y) \to 0.$$

Comparing the two yields the required base change isomorphism. Now if y is not a closed point of Y, we can consider the Cartesian diagram as in 4 where $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}$. Then u is flat and y is a closed point of Y' and we can reduce to this case by applying the following.

Lemma 2. *(Flat base change) Consider the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{4}$$

where f is qcqs [2] and u is flat and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the base change morphism

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

is an isomorphism for all $i \geq 0$.

Now when Y is a reduced local ring, a module M is free if and only if $\dim M_y$ is independent of y for each $y \in Y$ so using the now proven base change isomorphism

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) = H^0(X_y, \mathcal{F}(d)|_{X_y})$$

we obtain that $H^0(X, \mathcal{F}(d))$ is a finite free A-module if and only if $P_{\mathcal{F}|_{X_y}}(d)$ is independent of $y \in Y$.

□

Remark 2. *As a corollary, we obtain the valuative criterion for flatness in the case of a projective morphism since the constancy of the Hilbert polynomial can be checked after pulling back to a regular curve.*

Remark 3. *The Hilbert polynomial encodes a lot of geometric information about a projective variety X such as the dimension, degree of projective dimension, and arithmetic genus. In particular, these invariants are constant in projective flat families.*

[2] quasi-compact quasi-separated