Lecture 5: Base change, the Hilbert functor

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1 Remarks on base change

Last time we proved the constancy of Hilbert polynomials in projective flat families:

Theorem 1. *Let $f : X \to Y$ be a projective morphism over a locally Noetherian scheme Y. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $P_{\mathcal{F}|_{X_y}}(d)$ is locally constant for $y \in Y$. If Y is reduced, then the converse holds.*

In the process we proved the lemma that when $Y = \text{Spec } A$ is the spectrum of a Noetherian local ring, then $\mathcal{F}$ is flat if and only if $H^0(X, \mathcal{F}(d))$ is a finite free A-module for $d \gg 0$. Note that this statement immediately globalizes:

Corollary 1. *Let $f : X \to Y$ and $\mathcal{F}$ be as above with Y Noetherian. Then $\mathcal{F}$ is flat over Y if and only if $f_*\mathcal{F}(d)$ is a finite rank locally free sheaf for all $d \gg 0$.*

Then we had to use two base change results. Namely we needed to show the following isomorphism (still in the local case $Y = \text{Spec } A$):

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) \cong H^0(X_y, \mathcal{F}(d)_y) \tag{1}$$

for all $y \in Y$ and $d \gg 0$. In proving (1) we needed the following flat base change.

Lemma 1. *(Flat base change) Consider the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{2}$$

where f is qcqs [1] and u is flat and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the base change morphism

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

is an isomorphism for all $i \geq 0$.

[1] quasi-compact quasi-separated, though for our use separated suffices

Proof. (Sketch) The question is local on Y and Y' so we can assume that $Y = \text{Spec } A$ and $Y' = \text{Spec } B$ where B is a flat A-algebra. Then the higher direct image functors are just taking cohomology so the statement becomes that the natural map

$$H^i(X, \mathcal{F}) \otimes_A B \to H^i(X', u'^* \mathcal{F})$$

is an isomorphism of B-modules. When f is separated we can cover X by affines and compute $H^i(X, \mathcal{F})$ using Čech cohomology. Furthermore, the pullback of this open cover to X' is a cover of X' by affines from which we can compute $H^i(X', u'^* \mathcal{F})$. Now we use that tensoring by B preserves the cohomology of the Čech complex since B is flat. In the more general qcqs setting, one must use the Čech-to-derived spectral sequence.

□

We also noted that the proof of (1) did not actually use flatness of $\mathcal{F}$ over Y since it dealt with only global sections. Indeed we have the following more general base change without flatness.

Proposition 1. *(Base change without flatness) Suppose we have a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{3}$$

where f is projective, Y' and Y are Noetherian, and suppose $\mathcal{F}$ is a coherent sheaf on X. Then the base change morphism

$$u^* f_*(\mathcal{F}(d)) \to f'_* u'^*(\mathcal{F}(d))$$

is an isomorphism for $d \gg 0$.

Proof. (Sketch) The strategy is the same as in the proof of (1). First the question is local on Y so we can suppose $Y = \text{Spec } A$ is affine. Then we reduce to the case $Y' = \text{Spec } A'$ is affine using flat base change. Furthermore, we can suppose $X = \mathbb{P}^n_A$. Then we take a resolution $P_1 \to P_0 \to \mathcal{F} \to 0$ by direct sums of twisting sheaves $\mathcal{O}_X(a)$. Pulling back by u' gives us a resolution of $P'_1 \to P'_0 \to u'^* \mathcal{F} \to 0$ by direct sums of the corresponding twisting sheaves on X'. After twisting by $\mathcal{O}_X(d)$ (resp. $\mathcal{O}_{X'}(d)$) for $d \gg 0$, higher cohomologies vanish and so applying H^0 gives us a resolution $H^0(X, \mathcal{F}(d))$ as an A-module and a resolution of $H^0(X', u'^* \mathcal{F}(d))$ as an A' module by direct sums of $H^0(X, \mathcal{O}_X(a))$ (resp. $H^0(X', \mathcal{O}_{X'}(a))$). By identifying the spaces of sections of $\mathcal{O}_X(a)$ with degree a polynomials over A, it is clear that base change holds for this module:

$$H^0(X, \mathcal{O}_X(a)) \otimes_A A' \cong H^0(X', \mathcal{O}_{X'}(a)). \tag{4}$$

Applying $- \otimes_A A'$ to the resolution of $H^0(X, \mathcal{F}(d))$ yields a resolution of $H^0(X, \mathcal{F}(d)) \otimes_A A'$ and we see by 4 this is the same as the resolution of $H^0(X', u'^* \mathcal{F}(d))$. Since the base change morphisms for $\mathcal{O}_X(a)$ commute with those for $\mathcal{F}(d)$ we conclude the base change morphism for $\mathcal{F}(d)$ is an isomorphism. □

Here is an example to show that in general, even flatness of $\mathcal{F}$ is not enough to ensure that the base change morphism is an isomorphism.

Example 1. *Let $X = E \times_k E$ where E is an elliptic curve over a field k with origin $e \in E(k)$. Let $\Delta \subset X$ denote the diagonal and consider the line bundle*

$$L = \mathcal{O}_X(\Delta - p_2^* e)$$

where $p_i : X \to E$ are the projections. Now consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

where $Y = E$, $f = p_1$, $Y' = \operatorname{Spec} k$, and $u = e : \operatorname{Spec} k \to E$ is the origin. Then $X' = X_e \cong E$ and $f' \to \operatorname{Spec} k$ is just the structure map. The pullback $u'^* L = L|_{E_e} \cong \mathcal{O}_E$ so

$$f'_* u'^* L = H^0(E, \mathcal{O}_X) = k.$$

On the other hand, $f_* L$ is a torsion free sheaf on the integral regular curve Y so it is locally free. We may compute its stalk at the generic point of Y by flat base change. We get that $f_* L_\eta = H^0(E_\eta, \mathcal{O}_{E_\eta}(\Delta_\eta - e_\eta)) = 0$ since Δ_η and e_η are distinct points of the genus one curve E_η. Thus $f_* L = 0$ so $u^* f_* L = 0$ and we see that the base change map is not an isomorphism. Of course in this case, the projection f is flat and L is a line bundle so L is flat over Y.

What goes wrong here is that the cohomology of the fibers jumps at $e \in E$. This situation is completely understood by the following two theorems.

Theorem 2. *(Semi-continuity) Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X flat over Y. Then the function*

$$y \mapsto \dim H^i(X_y, \mathcal{F}_y)$$

is upper semi-continuous. Moreover, the function

$$y \mapsto \chi(X_y, \mathcal{F}_y) = \sum (-1)^i \dim H^i(X_y, \mathcal{F}_y)$$

is locally constant.

Theorem 3. *(Cohomology and base change) Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X flat over Y. Suppose for some $y \in Y$, the base change map*

$$\varphi^i_y : (R^i f_* \mathcal{F})_y \to H^i(X_y, \mathcal{F}_y)$$

is surjective. Then

1. *there exists an open neighborhood U of y such that for all $y' \in U$, $\varphi^i_{y'}$ is an isomorphism, and*
2. *φ^{i-1}_y is surjective if and only if $R^i f_* \mathcal{F}$ is locally free in a neighborhood of y.*

Often times in moduli theory, one needs to show that various constructions on families are functorial so that they induce a construction on the moduli space. Functoriality usually means compatibility with base change. As such, the following generalization (and direct corollary) of the cohomology and base change theorem is very useful.

Proposition 2. *Let $f : X \to Y$ and $\mathcal{F}$ be as above. Suppose that φ^i_y is an isomorphism and $R^i f_* \mathcal{F}$ is locally free (or equivalently φ^{i-1}_y is an isomorphism) for all $y \in Y$. Then for any locally Noetherian scheme Y' and cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

the base change map

$$\varphi^i_u : u^* R^i f_* \mathcal{F} \to R^i f'_* (u'^* \mathcal{F})$$

is an isomorphism. In particular, if $H^1(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, then $f_* \mathcal{F}$ is locally free and $u^* f_* \mathcal{F} \cong f'_* u'^* \mathcal{F}$.

When the conclusion of the proposition holds, we often say the formation of $R^i f_* \mathcal{F}$ commutes with arbitrary base change.

We won't prove semi-continuity and cohomology and base change here but let us say a few words about the proof. First, the statements are all local on Y so we may suppose $Y = \operatorname{Spec} A$ where A is local and Noetherian. The proofs then are based on the idea of Grothendieck to consider the functor on the category of A-modules given by

$$M \mapsto H^i(X, \mathcal{F} \otimes_A M).$$

Then one proves a sort of "representability" result for this functor. There exists a complex $K^\bullet$, the Grothendieck complex of $\mathcal{F}$, such that each term K^i is a finite free module, and such that there are isomorphisms

$$H^i(X, \mathcal{F} \otimes_A M) \cong H^i(K^\bullet \otimes_A M)$$

functorial in M. This reduces base change and semi-continuity problems to linear algebra of this complex $K^\bullet$ and the theorems follow from a careful study of the properties of complexes of flat modules under base change using Nakayama's lemma.

2 The Hilbert and Quot functors

Now we can define Hilbert functor of a projective morphism $f : X \to S$. Note that implicit in this is a fixed embedding of X into $\mathbb{P}^n_S$ and thus a fixed very ample line bundle $\mathcal{O}_X(1)$ that we can take the Hilbert polynomial with respect to.

Definition 1. *Let $f : X \to S$ be a projective morphism. The Hilbert functor $H_{X/S} : Sch_S \to Set$ is the functor*

$$T \mapsto \{\textit{closed subschemes } Z \subset X_T := X \times_S T \mid Z \to T \textit{ is flat and proper}\}.$$

This is a functor by pulling back Z along $T' \to T$. An element $(Z \subset X_T) \in H_{X/S}(T)$ is called a flat family of subschemes of X parametrized by T. Let P be any polynomial. We define the subfunctor $H^P_{X/S} \subset H_{X/S}$ by

$$H^P_{X/S}(T) = \{\textit{flat families of subschemes } Z \subset X_T \mid P_{Z_t}(n) = P(n) \textit{ for all } t \in T\}.$$

By the local constancy of Hilbert polynomials in flat projective families, we see that

$$H_{X/S} = \bigsqcup_P H^P_{X/S}$$

Our goal for the next few classes is to prove that for each $f : X \to S$ and P as above, $H^P_{X/S}$ is representable by a projective scheme, the *Hilbert scheme* $\mathrm{Hilb}^P_{X/S}$, over S.