Lecture 6: The Hilbert and Quot schemes

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Recall last time we defined the Hilbert functor $H_{X/S}$ parametrizing flat families of closed subschemes $Z \subset X_T$ over any base scheme T. Moreover, we noted that

$$H_{X/S} = \bigsqcup H^P_{X/S}$$

where P runs over all numerical polynomials and $H^P_{X/S}$ is the subfunctor parametrizing those flat families of closed subschemes for which the Hilbert polynomial $P_{Z_t}(d) = P$ for all $t \in T$.[1]

Giving a subscheme $i : Z \subset X_T$ is the same as giving an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{X_T}$ with quotient $i_*\mathcal{O}_Z$. We have that $Z \to T$ is flat if and only if $i_*\mathcal{O}_Z$ is flat over T, and $i_*\mathcal{O}_Z$ is a quotient of $\mathcal{O}_{X_T}$ with kernel $\mathcal{I}_Z$ so the Hilbert functor is the same as the functor for equivalence classes of quotients $q : \mathcal{O}_{X_T} \to \mathcal{F} \to 0$ where $\mathcal{F}$ is flat over T with Hilbert polynomial P. Two such quotients $(q, \mathcal{F})$ and $(q', \mathcal{F}')$ give the same ideal sheaf (and thus the same subscheme of X_T) if and only if there is an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}_{X_T} & \xrightarrow{q} & \mathcal{F} \\ \Vert & & \downarrow{\scriptstyle \alpha} \\ \mathcal{O}_{X_T} & \xrightarrow[q']{} & \mathcal{F}' \end{array}$$

Thus we have the equality of functors

$$H^P_{X/S}(T) = \{q : \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ flat over } T \text{ with proper support, } P_{\mathcal{F}_t}(n) = P(n)\}/\sim$$

where $\sim$ is the equivalence relation of pairs $(q, \mathcal{F})$ given by diagrams as above.

More generally, we can consider a fixed coherent sheaf $\mathcal{E}$ on X. For any $\varphi : T \to S$, let us denote by $\mathcal{E}_T$ the pullback of $\mathcal{E}$ to X_T or $\varphi^*\mathcal{E}$.

Definition 1. *The Quot functor $Q_{\mathcal{E},X/S} : Sch_S \to Set$ is the functor*

$$T \mapsto \{q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \textit{ flat over } T,\ \mathrm{Supp}(\mathcal{F}) \to T \textit{ is proper}\}/\sim$$

[1] We had a question as to why this disjoint union decomposition holds in general since we only showed the local constancy of Hilbert polynomials over a locally Noetherian base. Note however that for any $T \to S$, the morphism $X_T \to T$ is locally of finite presentation. For the local constancy we may restrict to $T = \operatorname{Spec} A$ being local in which case $X_T \to T$ is a morphism of finite presentation. Thus for any closed subscheme $Z \subset X_T$ flat and proper over T, $Z \to T$ is a morphism of finite presentation. Then a usual trick shows that $Z \to T$ is pulled back from a morphism $Z' \to T'$ where T' is finitely presented over S. Then we have constancy of the Hilbert polynomial for $Z' \to T'$ which implies constancy for $Z \to T$.

where $(q, \mathcal{F}) \sim (q', \mathcal{F}')$ *if and only if there exists an isomorphism* $\alpha : \mathcal{F} \to \mathcal{F}'$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{E}_T & \xrightarrow{q} & \mathcal{F} \\ \| & & \downarrow \alpha \\ \mathcal{E}_T & \xrightarrow[q']{} & \mathcal{F}' \end{array}$$

Given a polynomial P*, we have the subfunctor* $Q^P_{\mathcal{E},X/S}$ *of those quotients* $(q, \mathcal{F})$ *such that for each* $t \in T$*,* $P_{\mathcal{F}_t}(n) = P(n)$*. This is a functor by pullback, where we note that for any* $\varphi : T' \to T$*,* $\varphi^* q : \varphi^* \mathcal{E}_T = \mathcal{E}_{T'} \to \varphi^* \mathcal{F}$ *is surjective.*

Remark 1. *Note that* $H_{X/S} = Q_{\mathcal{O}_X, X/S}$ *by the above discussion.*

As before we have that

$$Q_{\mathcal{E},X/S} = \bigsqcup_P Q^P_{\mathcal{E},X/S}.$$

We have the following main representability result.

Theorem 1. *Let* $f : X \to S$ *be a projective morphism over a Notherian scheme* S *and let* P *be a polynomial. Then there exists a projective* S*-scheme* $\mathrm{Hilb}^P_{X/S}$ *as well as a closed subscheme*

$$\mathcal{Z}^P_{X/S} \subset X \times_S \mathrm{Hilb}^P_{X/S}$$

such that $\mathcal{Z}^P_{X/S} \to \mathrm{Hilb}^P_{X/S}$ *is flat and proper with Hilbert polynomial* P *and the pair* $(\mathrm{Hilb}^P_{X/S}, Z^P_{X/S})$ *represents the Hilbert functor* $H^P_{X/S}$*. More generally, if* $\mathcal{E}$ *is a coherent sheaf on* X*, then there exists a projective* S*-scheme* $\mathrm{Quot}^P_{\mathcal{E},X/S}$ *as well as a quotient sheaf*

$$q^P_{\mathcal{E},X/S} : \mathcal{E}_{\mathrm{Quot}^P_{\mathcal{E},X/S}} \to \mathcal{F}^P_{\mathcal{E},X/S} \to 0$$

on $\mathrm{Quot}^P_{\mathcal{E},X/S} \times_S X$ *which is flat with proper support over* $\mathrm{Quot}^P_{\mathcal{E},X/S}$ *and has Hilbert polynomial* P *such that the pair*

$$(\mathrm{Quot}^P_{\mathcal{E},X/S}, q^P_{\mathcal{E},X/S})$$

represents the Quot functor $Q^P_{\mathcal{E},X/S}$*.*

Thus we have projective fine moduli spaces $\mathrm{Hilb}^P_{X/S}$ and $\mathrm{Quot}^P_{\mathcal{E},X/S}$ for closed subschemes and quotients of a coherent sheaf respectively!

The basic idea of the construction is simple. To illustrate it, let us consider the Hilbert functor for $X = \mathbb{P}^n_k$ over a base field $S = \operatorname{Spec} k$. Now a subscheme $Z \subset X$ is determined by its equations which form an ideal $I \subset k[x_0, \ldots, x_n]$ which we can view as a linear subspace of the infinite dimensional vector space $k[x_0, \ldots, x_n]$. This gives, at least set theoretically, an inclusion

$$\{\text{Subschemes of } \mathbb{P}^n_k\} \hookrightarrow Gr(k[x_0, \ldots, x_n])$$

to some infinite dimensional Grassmannian of the vector space $k[x_0, \ldots, x_n]$. Now to proceed we need to do two things:

(a) cut down the dimension of the target to a finite dimensional Grassmannian which we proved already is representable by a projective scheme;

(b) show that the image of this set theoretic map is actually an algebraic subscheme which represents the functor of flat families.

These two steps respectively require the following two technical results.

Theorem 2. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf of $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m}$ with Hilbert polynomial P we have the following. For any $d \geq N$,*

1. *$H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ for all $i \geq 1$,*
2. *$\mathcal{F}(d)$ is generated by global sections, and*
3. *$H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ is surjective.*

Theorem 3. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \coprod_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

In the second theorem, we can think of S_P as well as the pullback $i_P^*\mathcal{F}$ which is necessarily flat over S_P as the fine moduli space for the functor that takes a scheme T to the set of morphisms to S which pull back $\mathcal{F}$ to be flat with Hilbert polynomial P.

1 Proof of the representability of $H^P_{X/S}$ and $Q^P_{\mathcal{E},X/S}$

We are now going to prove the representability of $H^P_{X/S}$ and $Q^P_{\mathcal{E},X/S}$ assuming uniform CM regularity and flattening stratifications. We will return to the proofs of these statements later. Note that $H^P_{X/S} = Q^P_{\mathcal{O}_X,X/S}$ so we will prove representability of $Q^P_{\mathcal{O}_X,X/S}$ for any coherent sheaf $\mathcal{F}$ on X.

Proof. **Step 1:** First we reduce to the case $X = \mathbb{P}^n_S$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. This is a consequence of the following lemmas.

Lemma 1. *For any integer r, tensoring by $\mathcal{O}_{X_T}(r)$ induces an isomorphism of functors*

$$Q^{P(d)}_{\mathcal{E},X/S} \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}.$$

Proof. Tensoring by a line bundle is an equivalence of categories since there is an inverse given by tensoring by the dual. Thus for each, T, tensoring by $\mathcal{O}_{X_T}(r)$ induces a bijection

$$Q^{P(d)}_{\mathcal{E},X/S}(T) \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}(T).$$

This is a natural transformation since for any $\varphi : T' \to T$, $\varphi^*\mathcal{O}_{X_T}(r) = \mathcal{O}_{X_{T'}}(r)$. □

Lemma 2. *Suppose $\alpha : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ is a quotient of coherent sheaves on X. Then the induced map $Q^P_{\mathcal{E},X/S} \to Q^P_{\mathcal{E}',X/S}$ is a closed subfunctor.*

Proof. This map is given by noting that a quotient $q : \mathcal{E} \twoheadrightarrow \mathcal{F}$ induces a quotient $q' = q \circ \alpha : \mathcal{E}' \twoheadrightarrow \mathcal{F}$ by composition. We need to show that for any scheme T' over S and object $(q', \mathcal{F}) \in Q^P_{\mathcal{E}',X/S}(T')$, there exists a closed subscheme $T \subset T'$ satisfying the following universal property. For any other S-scheme T'', a morphism $\varphi : T'' \to T'$ factors through T if and only if $\varphi^* q' : \mathcal{E}'_{T''} \twoheadrightarrow \varphi^*\mathcal{F}$ factors through a map $q : \mathcal{E}_{T''} \to \varphi^*\mathcal{F}$. Since the morphism $X_{T'} \to T'$ is of locally of finite presentation, and the condition of being closed can be checked locally, we may assume T' is affine in which case $X_{T'} \to T'$ is of finite presentation and then we can use the finite presentation trick to reduce to the case that T', T'' are Noetherian.

Let $\mathcal{K} = \ker(\alpha) \subset \mathcal{E}'$ be the kernel of α. Then the morphism $q' : \mathcal{E}'_{T'} \to \mathcal{F}$ factors through $\mathcal{E}_{T'}$ if and only if the composition $\mathcal{K}_{T'} \to \mathcal{F}$ is the zero map. Indeed $\mathcal{K}_{T'}$ surjects onto the kernel of $\mathcal{E}'_{T'} \to \mathcal{E}_{T'}$ by right exactness of pullback and a morphism factors through a surjection if and only if it is 0 on the kernel. Let us denote the composition $\mathcal{K}_{T'} \to \mathcal{F}$ by r. Thus we want to show that there exists a closed subscheme $T \subset T'$ such that a morphism $\varphi : T'' \to T'$ factors through T if and only if the composition $\varphi^* r : \mathcal{K}_{T''} \to \varphi^*\mathcal{F}$ of coherent sheaves on $X_{T''}$ is zero. The result now follows by applying the following lemma.

□

Lemma 3. *Let $f : X \to S$ be a projective morphism over a Noetherian scheme and let $r : \mathcal{K} \to \mathcal{F}$ be a map of coherent sheaves on X with $\mathcal{F}$ flat over S. Then there exists a closed subscheme $Z \subset S$ such that for any T Noetherian and any $\alpha : T \to S$, α factors through Z if and only if $\alpha^* r$ is the zero map.*

Proof. For any d, r is zero if and only if the twist $r(d) : \mathcal{K}(d) \to \mathcal{F}(d)$ is zero. For large enough $d \gg 0$, the pushforward $f_*\mathcal{F}(d)$ is locally free since $\mathcal{F}$ is flat over S and $\mathcal{K}(d)$ is gloablly generated over S so that

$$f^* f_* \mathcal{K}(d) \to \mathcal{K}(d)$$

is surjective. Thus $r(d)$ is 0 if and only if $f^* f_* \mathcal{K}(d) \to \mathcal{F}(d)$ is 0 if and only if $f_* r(d) : f_*\mathcal{K}(d) \to f_*\mathcal{F}(d)$ is 0. By the hom-tensor adjunction, using that $f_*\mathcal{F}(d)$ is locally free, this is the same as the cosection $t_d : f_*\mathcal{K}(d) \otimes (f_*\mathcal{F}(d))^\vee \to \mathcal{O}_S$ vanishing. Now the cosection t_d defines an ideal sheaf $I_d \subset \mathcal{O}_S$ by its image and it is clear that t_d vanishes at a point $s \in S$ if and only if $s \in V(I_d) = Z_d$.

Now consider the chain

$$I_{d_0} \subset I_{d_0} + I_{d_0+1} \subset \dots$$

where d_0 is some large enough number so that $\mathcal{K}(d)$ is globally generated and $f_*\mathcal{F}(d)$ is locally free. By the Noetherian condition, this chain terminates in some ideal I with vanishing subscheme $V(I) = Z$ the scheme theoretic intersection of the Z_d. Now for large enough $d \gg 0$, the formation of $f_*\mathcal{K}(d)$ and $f_*\mathcal{F}(d)$ commute with base change so $s \in Z$ if and only if $(t_d)_s = 0$ for all $d \gg 0$ if and only if $r(d)_s = 0$ for all $d \gg 0$ if and only if $r_s = 0$.[2]

We will check this Z satisfies the universal property. Suppose $\alpha : T \to S$ satisfies that $\alpha^* r$ is the zero map, then $\alpha^* r(d)$ is the zero map for $d \gg 0$ and $f_*\mathcal{K}(d)$, $f_*\mathcal{F}(d)$ commute with base change by α for d large enough (depending on α) so $\alpha^* f_* r(d) : \alpha^* f_*\mathcal{K}(d) \to \alpha^* f_*\mathcal{F}(d)$

[2]In particular, all the Z_d have the same set theoretic support as Z and we only need to worry about the right scheme structure.

is the zero map so α factors through Z_d for all $d \gg 0$ so α factors through Z. On the other hand, if α factors through Z, then for all $d \gg 0$, $\alpha^* f_* \mathcal{K}(d) \to \alpha^* f_* \mathcal{F}(d)$ is the zero map but by base change without flatness, for large enough d, the formation of these pushforwards commutes with basechange by α so we have that

$$(f_T)_* \alpha^* \mathcal{K}(d) \to (f_T)_* \alpha^* \mathcal{F}(d)$$

is the zero map. Thus $f_T^*(f_T)_* \alpha^* \mathcal{K}(d) \to \alpha^* \mathcal{F}(d)$ is the zero map. Since $\alpha^* \mathcal{K}(d)$ is globally generated for $d \gg 0$, then $\alpha^* r(d) : \alpha^* \mathcal{K}(d) \to \alpha * \mathcal{F}(d)$ is the zero map for d large enough so $\alpha^* r$ is the zero map.

$\square$

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