Lecture 7: The Hilbert and Quot schemes (cont.)

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1 Proof of representability (cont.)

We are continuing the proof of representability of the Quot functor $Q^P_{\mathcal{E},X/S}$ (and thus Hilbert functors) for $\mathcal{E}$ a coherent sheaf on X with $X \to S$ projective over S Noetherian.

Proof. **Step 1:** Recall we are reducing to the case $X = \mathbb{P}^n$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. We have completed the proof of the following two lemmas.

Lemma 1. *For any integer r, tensoring by $\mathcal{O}_{X_T}(r)$ induces an isomorphism of functors*

$$Q^{P(d)}_{\mathcal{E},X/S} \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}.$$

Lemma 2. *Suppose $\alpha : \mathcal{E}' \to \mathcal{E}$ is a quotient of coherent sheaves on X. Then the induced map $Q^P_{\mathcal{E},X/S} \to Q^P_{\mathcal{E}',X/S}$ is a closed subfunctor.*

Now given any sheaf $\mathcal{E}$ on X and $i : X \to \mathbb{P}^n_S$ the projective embedding, a quotient $q : \mathcal{E} \to \mathcal{F}$ is the same as a quotient $i_*\mathcal{E} \to i_*\mathcal{F}$ since i_* is an equivalence of categories between sheaves on X and sheaves on $\mathbb{P}^n_S$ supported on X which preserves Hilbert polynomials. Thus suppose $X = \mathbb{P}^n_S$. Then for $a \gg 0$, $\mathcal{E}(a)$ is globally generated so there is a surjection

$$\mathcal{O}_X(a)^{\oplus k} \to \mathcal{E}(a)$$

for some k. Thus by the second lemma above,

$$Q^P_{\mathcal{E}(a),X/S} \hookrightarrow Q^P_{\mathcal{O}_X(a)^{\oplus k},X/S}$$

is a closed subfunctor so it suffices to prove $Q^P_{\mathcal{O}_X(a)^{\oplus k},X/S}$ is representable by a projective scheme over S. Then by the first lemma, there is an isomorphism of functors

$$Q^{P(d-a)}_{\mathcal{O}_X^{\oplus k},X/S} \cong Q^{P(d)}_{\mathcal{O}_X^{\oplus k}(a),X/S}.$$

Step 2: Now we are in the situation $X = \mathbb{P}^n_S$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. Let $q : \mathcal{E}_T \to \mathcal{F}$ be an element of $Q^P_{\mathcal{E},X/S}(T)$ and let $\mathcal{K}$ be the kernel of q. By flatness of $\mathcal{F}$ over T, for any $t \in T$ we have that

$$0 \to \mathcal{K}_t \to \mathcal{E}_t \to \mathcal{F}_t \to 0$$

is exact. Then by additivity of Euler characteristics, the Hilbert polynomial $P_{\mathcal{K}_t}$ is given by

$$P_{\mathcal{K}_t}(d) = k\binom{n+d}{d} - P(d).$$

In particular it is independent of $t \in T$ or even of $(q, \mathcal{F})$. Thus by uniform CM regularity applied to $\mathcal{K}_t$ and $\mathcal{E}_t$ [1], there exists an N depending only on $P(d)$, k and n such that for all T, all $(q, F) \in Q^P_{\mathcal{E},X/S}(T)$, and all $t \in T$, we have that for all $a \geq N$,

- $H^i(X_t, \mathcal{K}_t(a)) = H^i(X_t, \mathcal{E}_t(a)) = H^i(X_t, \mathcal{F}_t(a)) = 0$ for $i \geq 1$, and
- $\mathcal{K}_t(a), \mathcal{E}_t(a)$ and $\mathcal{F}_t(a)$ are globally generated.

Since $\mathcal{E}$ and $\mathcal{F}$ are both flat, then $\mathcal{K}$ is also flat. Then we can apply cohomology and base change to see that for all $d \geq N$,

$$0 \to (f_T)_*\mathcal{K}(a) \to (f_T)_*\mathcal{E}(a) \to (f_T)_*\mathcal{F}(a) \to 0 \tag{1}$$

is an exact sequence of locally free sheaves of rank

$$k\binom{n+a}{a} - P(a),\ k\binom{n+a}{a}, \text{ and } P(a)$$

respectively. Moreover, the formation of all three of these locally free sheaves is compatible with base change by any $T' \to T$ with T' locally Noetherian.[2] The sequence (1) gives an element of $Gr(P(a), f_*\mathcal{E}(a))(T)$ and since (1) is compatible with base change, the induced set theoretic map

$$Q^P_{\mathcal{E},X/S}(T) \to Gr(P(a), f_*\mathcal{E}(a))(T)$$

is a natural transformation of functors.

Moreover, one can compute explicitly that $f_*\mathcal{E}(a)$ is actually the free sheaf of rank $k\binom{n+a}{a}$. Indeed, $\mathcal{E}(a) = \mathcal{O}_X(a)^{\oplus k}$ so it suffices to check for $k = 1$ in which case $f_*\mathcal{O}_X(a)$ restricted to any affine open Spec $A \subset S$ is simply

$$A[x_0, \ldots, x_n]_a$$

so globally $f_*\mathcal{O}_X(a) = \mathcal{O}_S[x_0, \ldots, x_n]_a$ where $\mathbb{P}^n_S = \mathrm{Proj}_S \mathcal{O}_S[x_0, \ldots, x_n]$. Thus

$$Gr(P(a), f_*\mathcal{E}(a)) = Gr_S\left(P(a), k\binom{n+a}{a}\right)$$

which we showed previously is representable by a projective S-scheme. More canonically, $f_*\mathcal{O}_X(a)^{\oplus k} = V_a \otimes_{\mathbb{Z}} \mathcal{O}_S$ where $V_a = \mathbb{Z}[x_0, \ldots, x_n]_a^{\oplus k}$ so we can write the Grassmannian as $G_S = G \times_{\mathbb{Z}} S$ where $G = Gr(P(a), V_a)$.

Step 3: Our strategy now is to show that the natural transformation of functors $Q^P_{\mathcal{E},X/S} \to G_S$ is an inclusion of a subfunctor and then identify the subfunctor $Q^P_{\mathcal{E},X/S}$ with the functor of points of some locally closed subvariety of G_S.

Toward that end, we need to show that for T and any $(q, \mathcal{F}) \in Q^P_{\mathcal{E},X/S}(T)$, $(q, \mathcal{F})$ is determined by the sequence (1):

$$0 \to (f_T)_*\mathcal{K}(a) \to (f_T)_*\mathcal{E}(a) \to (f_T)_*\mathcal{F}(a) \to 0.$$

[1] as well as a diagram chase to conclude the result for $\mathcal{F}_t$

[2] I'll leave it to the reader to use the finite presentation trick and convince themselves that this is good enough!

By global generation and the fact that these sheaves are all locally free, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_T^*(f_T)_*\mathcal{K}(a) & \longrightarrow & f_T^*(f_T)_*\mathcal{E}(a) & \longrightarrow & f_T^*(f_T)_*\mathcal{F}(a) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}(a) & \longrightarrow & \mathcal{E}(a) & \xrightarrow{q(a)} & \mathcal{F}(a) & \longrightarrow & 0 \end{array}$$

where the vertical maps are surjections and the horizontal sequences are exact. Let

$$h : f_T^*(f_T)_*\mathcal{K}(a) \to \mathcal{E}(a)$$

be the composition. It suffices to show that q may be determined from h, but indeed by exactness, the cokernel of h is natural identified with $q(a) : \mathcal{E}(a) \to \mathcal{F}(a)$ and so by twisting by $\mathcal{O}_X(-a)$ we recover q from the sequence (1) and conclude that

$$Q^P_{\mathcal{E},X/S}(T) \to G_S(T)$$

is injective.

Step 4: Now we will use flattening stratifications to show that $Q^P_{\mathcal{E},X/S}$ as a subfunctor of the Grassmannian is representable by a locally closed subscheme. Over G_S we have the universal quotient sequence

$$0 \to \mathcal{K} \to V_a \otimes \mathcal{O}_{G_S} \to \mathcal{Q} \to 0$$

and $V_a \otimes \mathcal{O}_{G_S} = (f_{G_S})_*\mathcal{E}_{G_S}(a)$ where $f_{G_S} : \mathbb{P}^n_{G_S} \to G_S$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_{G_S}}(d)^{\oplus k}$.

Now we can pullback this sequence to get

$$0 \to f^*_{G_S}\mathcal{K} \to f^*_{G_S}(f_{G_S})_*\mathcal{E}_{G_S}(a) \to f^*_{G_S}\mathcal{Q} \to 0.$$

The middle sheaf comes with a surjective map to $\mathcal{E}_{G_S}(a)$ since $\mathcal{E}_{G_S}(a)$ is gloablly generated on any projective space ($a > 0$). Let us denote by $h : f^*_{G_S}\mathcal{K} \to \mathcal{E}_{G_S}(a)$ the composition and let $\mathcal{F}$ be the cokernel of h. Then $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_{G_S}$ and we can consider the flattening stratification for $\mathcal{F}$ over G_S.

Let $G_S^{P(d+a)} \subset G_S$ be the stratum over which $\mathcal{F}$ is flat with Hilbert polynomial $P(d+a)$. Then $G_S^{P(d+a)}$ is universal for maps to $T \to G_S$ such that $\mathcal{F}_T$ is flat over T with Hilbert polynomial $P(d+a)$, but that exactly means that the quotient map $\mathcal{E}_T \to \mathcal{F}_T(-a)$ is an element of the subfunctor $Q^P_{\mathcal{E},X/S}$ so the locally closed subscheme $G_S^{P(d+a)}$ with the restriction of the quotient map $\mathcal{E}_{G_S} \to \mathcal{F}$ represents the subfunctor $Q^P_{\mathcal{E},X/S}$.

Step 5: Finally we show that $Q^P_{\mathcal{E},X/S}$ satisfies the valuative criterion of properness. This implies the stratum $G_S^{P(d+a)}$ is actually a closed subscheme of G_S. We showed earlier that G_S is projective over S so we conclude that $Q^P_{\mathcal{E},X/S}$ is representable by a projective scheme over S.

We already showed that the special case of the Hilbert functor is proper. The proof for $Q^P_{\mathcal{E},X/S}$ is similar. Let $T = \operatorname{Spec} R$ the spectrum of a DVR and let $T^0 = \operatorname{Spec} K$ the spectrum of its function field. We need to show that for any solid diagram as below, there exists a unique dashed arrow.

$$\begin{array}{ccc} T^0 & \longrightarrow & Q^P_{\mathcal{E},X/S} \\ \downarrow & \nearrow & \downarrow \\ T & \longrightarrow & S \end{array}$$

That is, given $(q^0, \mathcal{F}^0) \in Q^P_{\mathcal{E},X/S}(T^0)$, there is a unique extension to a flat quotient $(q, \mathcal{F})$. For this we can compose $\mathcal{E}_T \to \mathcal{E}_{T^0} \to \mathcal{F}^0$ and let $\mathcal{F}$ be the image of $\mathcal{E}_T$ with q the composition. Then $\mathcal{F}$ is flat and by the criterion for flatness over a DVR, it is the unique flat extension so this gives the required lift.

□

2 Some applications

Here we discuss some applications that follow from the representability and projectivity of Hilbert and Quot schemes.

2.1 Grassmannians of coherent sheaves

We consider the case that $f = id_S$ is the identity. In this case $\mathcal{E}$ is a coherent sheaf on S, flatness is equivalent to local freeness, and the Hilbert polynomial of $\mathcal{F}_s$ is just the dimension $\dim_{k(s)} \mathcal{F}_s$. Thus, given a k, we have the Quot scheme $Q^k_{\mathcal{E},S/S}$ for constant Hilbert polynomial $P(d) = k$ so that a map

$$T \to Q^k_{\mathcal{E},S/S}$$

is a locally free quotient $q : \mathcal{E}_T \to \mathcal{V}$ of rank k on T. When $\mathcal{E}$ is itself a locally free sheaf, this is just the Grassmannian $Gr_S(k, \mathcal{E})$ so we have Grassmannians for any coherent sheaf $\mathcal{E}$ which we also denote $Gr_S(k, \mathcal{E})$. In particular, when $k = 1$, we denote $Gr_S(1, \mathcal{E})$ by $\mathbb{P}(\mathcal{E})$ the projective "bundle" of $\mathcal{E}$ whose fiber over $s \in S$ is the projectivization of the vector space $\mathcal{E}_s$.

2.2 Lifting rational curves

Let X/k be a projective variety over a field. A degree r rational curve $C \subset X$ is a genus 0 curve with $\deg(\mathcal{O}_X(1)|_C) = r$. The Hilbert polynomial of such a curve is always $P(d) = rd + 1$. Indeed, $\chi(O_C) = 1$ since C is rational and $H^0(C, \mathcal{O}_C(d)) = rd + 1$ by Riemann-Roch for $d \gg 0$.

Now let $\mathcal{X}$ be a projective scheme over $\mathbb{Z}$ with geometric generic fiber $\mathcal{X}_{\overline{\mathbb{Q}}} =: X$ and $\mathcal{X}_{\mathbb{F}_p} =: X_p$. Suppose X_p contains a degree d rational curve for infinitely many d. Then X also contains a degree d rational curve. Indeed, consider $H^{rd+1}_{\mathcal{X}/\text{Spec }\mathbb{Z}}$. This is a projective scheme over Spec $\mathbb{Z}$ so it has finitely many irreducible components. Since the fiber of $H^{rd+1}_{X/\text{Spec }\mathbb{Z}}$ is nonempty over infinitely many Spec $\mathbb{F}_p$, there must be some component which dominates Spec $\mathbb{Z}$. Since the morphism to Spec $\mathbb{Z}$ is proper then the fiber over Spec $\mathbb{Q}$ must be nonempty. But a Spec $\overline{\mathbb{Q}}$ point of the fiber over Spec $\mathbb{Q}$ exactly corresponds to a rational curve of degree d on X.