Lecture 8: Hom schemes, CM regularity

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1 Hom schemes

Let X and Y be two schemes over S. The hom functor $\mathcal{H}om_S(X,Y) : Sch_S \to Set$ is given by

$$T \mapsto \{\text{morphisms } X_T \to Y_T \text{ over } T\}.$$

Theorem 1. *Suppose X and Y are projective over S with $X \to S$ flat. Then $\mathcal{H}om_S(X,Y)$ is representable by a quasi-projective scheme $\mathrm{Hom}_S(X,Y)$ over S.*

Proof. Given $f : X_T \to Y_T$, we have the graph $\Gamma_f : X_T \to X_T \times_T Y_T = (X \times_S Y)_T$ which is a closed embedding. Now $\mathrm{im}(\Gamma_f) \cong X_T$ is flat over T by assumption so it defines a map $T \to \mathrm{Hilb}_{(X\times_S Y)/S}$. This construction is compatible with basechange so we obtain a natural transformation of functors

$$\mathcal{H}om_S(X,Y) \to H_{(X\times_S Y)/S}.$$

Since a morphism is determined by its graph, this is a subfunctor. Moreover, we can characterize the graphs of morphisms as exactly those closed subschemes $Z \subset X_T \times_T Y_T$ such that the projection $Z \to X_T$ is an isomorphism. This identifies $\mathcal{H}om_S(X,Y)$ with the subfunctor of $H_{(X\times_S Y)/S}$ given by

$$T \mapsto \{\text{closed subsets } Z \subset X_T \times_T Y_T \mid Z \to T \text{ flat and proper, } Z \to X_T \text{ is an isomorphism}\}.$$

We will prove this is representable by an open subscheme of $\mathrm{Hilb}_{(X\times_S Y)/S}$.

We can consider the universal family $\mathcal{Z} \to \mathrm{Hilb}_{(X\times_S Y)/S}$ which is a closed subscheme of $X \times_S Y \times_S \mathrm{Hilb}_{(X\times_S Y)/S}$. Then $\mathcal{Z}$ comes with a projection $\pi : \mathcal{Z} \to X \times_S \mathrm{Hilb}_{(X\times_S Y)/S}$. Now we consider the diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\ \pi\ } & X \times_S \mathrm{Hilb}_{(X\times_S Y)/S} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathrm{Hilb}_{(X\times_S Y)/S} & = & \mathrm{Hilb}_{(X\times_S Y)/S}. \end{array}$$

Then the required open subscheme is given by the following lemma.

Proposition 1. *Let $T = \mathrm{Spec}\ R$ be the spectrum of a Noetherian local ring and let $0 \in T$ be the closed point. Let $f : X \to T$ be flat and proper and $g : Y \to T$ proper. Let $p : X \to Y$ be a morphism such that $p_0 : X_0 \to Y_0$ is an isomorphism. Then $p : X \to Y$ is an isomorphism.*

Proof. Since X is proper and Y is separated over T, the morphism $p : X \to Y$ must be proper. Moreover, since g is proper, every closed point of Y lies in Y_0. Furthermore, since p_0 is an isomorphism, then p has finite fibers over closed points of Y so p is quasi-finite. Indeed since p is proper, the fiber dimension is upper-semicontinuous on Y and it is 0 on closed points. Therefore r is finite, and in particular, affine. This implies that $R^i p_* \mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ and $i \geq 1$. Now the result follows if we know that f is flat. Indeed in this case, $p_* \mathcal{O}_X$ is locally free of rank one by cohomology and base change. On the other hand, the natural map $\mathcal{O}_Y \to p_* \mathcal{O}_X$ is an isomorphism at all closed points $y \in Y_0 \subset Y$ and since both source and target are line bundles, it must be an isomorphism. Then, since p is affine, we have

$$X = \mathrm{Spec}_Y f_* \mathcal{O}_X = \mathrm{Spec}_Y \mathcal{O}_Y = Y.$$

Thus, it suffices to prove the following that p is flat. We will use the following lemma.

Lemma 1. *Let $p : X \to Y$ be a morphism of locally Noetherian T-schemes over a locally Noetherian scheme T. Let $x \in X$ a point in the fiber X_t for $t \in T$ and set $y = p(x)$ its image in the fiber Y_t. Then the following are equivalent:*

1. *X is flat over T at x and $p_t : X_t \to Y_t$ is flat at $x \in X_t$;*
2. *Y is flat over T at y and p is flat at $x \in X$.*

Proof. Consider the sequence local ring homomorphisms

$$\mathcal{O}_{T,t} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

Let $I = \mathfrak{m}_t \mathcal{O}_{Y,y}$. Suppose (1) holds. Then $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{T,t}$ module and $\mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}/I$-module. Consider the composition

$$\mathfrak{m}_t \otimes \mathcal{O}_{X,x} \to I \otimes \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}.$$

The first map is surjective by right exactness of tensor products and the composition is injective by since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{T,t}$ so both maps are in fact injections. Thus

$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/I, \mathcal{O}_{X,x}) = 0. \tag{1}$$

Since $I \subset \mathfrak{m}_y$, then I annihilates the residue field $k(y)$ and one can check that Equation (1) and the assumptions imply that $\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(k(y), \mathcal{O}_{X,x}) = 0$ by the following lemma we will leave as an exercise.

Lemma 2. *Suppose R is a Noetherian ring and $I \subset R$ is a proper ideal. Let M be an R-module such that M/IM is a flat R/I-module and such that*

$$\mathrm{Tor}_1^R(R/I, M) = 0.$$

Then for any I-torsion R-module N,

$$\mathrm{Tor}_1^R(N, M) = 0.$$

Then $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module by the local criterion for flatness.

Since everything is local, $\mathcal{O}_{X,x}$ is in fact faithfully flat over $\mathcal{O}_{Y,y}$. Now we want to show that $\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0$. Pulling back the sequence

$$0 \to \mathfrak{m}_t \to \mathcal{O}_{T,t} \to k(t) \to 0.$$

to $\mathcal{O}_{Y,y}$ gives us

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \to \mathfrak{m}_t \otimes \mathcal{O}_{Y,y} \to I \to 0.$$

Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, then

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} \to \mathfrak{m}_t \otimes \mathcal{O}_{X,x} \to I \otimes \mathcal{O}_{X,x} \to 0.$$

We saw above that the second map is injective so

$$\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} = 0$$

but $\mathcal{O}_{X,x}$ is faithfully flat over $\mathcal{O}_{Y,y}$ so $\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0$ and $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{T,t}$.

For the converse, suppose (2) holds. Then $Y \to T$ is flat at $y \in Y$ and $p : X \to Y$ is flat at $x \in X$ so the composition $X \to T$ is flat at $x \in X$. Moreover, p_t is the pullback p to Y_t and flatness is stable under basechange so p_t is flat at $x \in X$.

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Corollary 1. *Let $f : X \to T$ be flat and proper and $g : Y \to T$ proper over a Noetherian scheme T. Let $p : X \to Y$ be a morphism. Then there exists an open subscheme $U \subset T$ such that for any T' and $\varphi : T' \to T$, φ factors through U if and only if $\varphi^* p : X_{T'} \to Y_{T'}$ is an isomorphism.*

Proof. The locus where $p : X \to Y$ is an isomorphism is open on the target Y so let $Z \subset Y$ be the closed subset over which p is not an isomorphism. Since g is proper, $g(Z) \subset T$ is closed. Let $U \subset T$ be its complement. By the proposition, a point $t \in T$ is contained in U if and only if the the map on the fibers $p_t : X_t \to Y_t$ is an isomorphism. Since this is a fiberwise condition on $t \in T$, it is clear that U satisfies the required universal property. □

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2 Castelnuovo-Mumford regularity

We will now discuss the first main ingredient in the proof of representability of Hilbert and Quot functors.

Theorem 2. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf of $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m}$ with Hilbert polynomial P we have the following. For any $d \geq N$,*

1. *$H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ for all $i \geq 1$,*

2. $\mathcal{F}(d)$ *is generated by global sections, and*

3. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective.*

To prove this we will define a more general notion of the Castelnuovo-Mumford regularity of a sheaf $\mathcal{F}$ on projective space.

Definition 1. *(CM regularity) A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is said to be m-regular if*

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0$$

for all $i > 0$.

The notion of CM regularity is well adapted to running inductive arguments by taking a hyperplane section.

Proposition 2. *Let $\mathcal{F}$ be m-regular. Then*

1. $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ *for all $d \geq m - i$ and $i > 0$, that is, $\mathcal{F}$ is m' regular for all $m' \geq m$,*

2. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective for all $d \geq m$.*

3. $\mathcal{F}(d)$ *is globally generated for all $d \geq m$, and*

Proof. The definition of m-regularity and the conclusions of the proposition can all be checked after passing to a field extension since field extensions are faithfully flat so we may suppose the field k is infinite. Now we will induct on the dimension n.

If $n = 0$ the results trivially hold since all higher cohomology vanishes, all sheaves are globally generated and $\mathcal{O}(1) = \mathcal{O}$. Suppose $n > 0$ and let $H \subset \mathbb{P}^n_k$ be a general hyperplane.[1] Now consider the short exact sequence

$$0 \to \mathcal{F}(m-i-1) \to \mathcal{F}(m-i) \to \mathcal{F}_H(m-i) \to 0$$

where $\mathcal{F}_H = \mathcal{F}|_H$ is the restriction. Taking the long exact sequence of cohomology yields

$$\ldots \to H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}_H(m-i)) \to H^{i+1}(\mathbb{P}^n_k, \mathcal{F}(m-i-1)) \to \ldots,$$

The first and last terms are 0 for all $i > 0$ by assumption so $H^i(H, \mathcal{F}_H(m-i)) = 0$ for all $i > 0$. That is, $\mathcal{F}_H$ is m-regular.

We will continue next time.

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[1]Here general means that H avoids all associated points of $\mathcal{F}$. This is where we use the infinite field assumption.