Lecture 9: CM regularity, flattening stratifications

10/07/2019

1 CM regularity (cont.)

Recall that a sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is m-regular is $H^i(\mathcal{F}(m-i)) = 0$ for all $i \geq 1$. We are proving the following.

Proposition 1. *Let $\mathcal{F}$ be m-regular. Then*

1. $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ *for all $d \geq m - i$ and $i > 0$, that is, $\mathcal{F}$ is m' regular for all $m' \geq m$,*
2. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective for all $d \geq m$.*
3. $\mathcal{F}(d)$ *is globally generated for all $d \geq m$, and*

Proof. Last time we started the proof by showing that for a general hyperplane H, the restriction $\mathcal{F}_H$ of $\mathcal{F}$ to H is an m-regular sheaf on the projective space H. Now by induction, the conclusions of the proposition hold for $\mathcal{F}_H$ since it is supported on the one dimension lower projective space H. Now we twist to obtain an exact sequence

$$\ldots \to H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}(m+1-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}(m+1-i)) \to \ldots.$$

Now the last term is zero by conclusion (1) applied to $\mathcal{F}_H$ and the first term is zero by assumption so the middle term is zero, i.e., $\mathcal{F}$ is $(m+1)$-regular. Now we induct on m to see it is m'-regular for all $m' \geq m$.[1] This proves (1).

Next, consider the diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) & \xrightarrow{\alpha} & H^0(H, \mathcal{F}_H(d)) \otimes H^0(H, \mathcal{O}_H(1)) \\ \downarrow \gamma & & \downarrow \delta \\ H^0(\mathbb{P}^n_k, \mathcal{F}(d+1)) & \xrightarrow[\beta]{} & H^0(H, \mathcal{F}_H(d+1)) \end{array}$$

where the horizontal maps are restriction to H and suppose $d \geq m$. Now the restriction $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \to H^0(H, \mathcal{F}_H(d))$ is surjective since $H^1(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ by conclusion (1) thus α is surjective. For the same reason, β is surjective. Moreover, δ is surjective by conclusion (2) for $\mathcal{F}_H$. Thus $\beta \circ \gamma$ is surjective but the kernel of β is exactly the image of $\gamma(- \otimes h)$:

[1] Note here we have used extensively that coherent cohomology is preserved by closed embeddings so that the cohomology of $\mathcal{F}_H$ on $\mathbb{P}^n_k$ is the same as that on H.

$H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ where $h \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ is the defining equation of H by the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{F}(d) \to \mathcal{F}(d+1) \to \mathcal{F}_H(d+1) \to 0$$

induced by multiplication by h. Thus $\ker(\beta)$ is contained in the image of γ so γ must be surjective and $\mathcal{F}$ satisfies (2).

Finally, the global generation of $\mathcal{F}(d)$ is equivalent to the fact that for each point $x \in \mathbb{P}^n_k$, there exists a collection of section $s_i \in H^0(\mathbb{P}^n_k, \mathcal{F}(d))$ such that $\bar{s}_i = s_i(x) \in \mathcal{F}(d) \otimes k(x)$ span the fiber $\mathcal{F}(d) \otimes k(x)$. By (2), we have a surjection

$$H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(a)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+a))$$

for all $a \geq 1$. For large enough $a \gg 0$, $\mathcal{F}(d+a)$ is globally generated by Serre vanishing so for each $x \in X$, there exists such sections $s_i \in H^0(\mathbb{P}^n_k, \mathcal{F}(d+a))$ whose values at x span the fiber, but since every such section comes from multiplying a section of $\mathcal{F}(d)$ by a homogeneous polynomial, there must be sections of $\mathcal{F}(d)$ spanning the fiber at x so $\mathcal{F}(d)$ is globally generated.

□

Corollary 1. *Suppose*

$$0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$$

is a short exact sequence of coherent sheaves. Suppose $\mathcal{F}''$ is $(m+1)$-regular and $\mathcal{F}$ is m-regular. Then $\mathcal{F}'$ is m-regular. In particular, all the three sheaves are in fact $(m+1)$-regular.

Proof. Consider the long exact sequence

$$\ldots \to H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}'(m-i)) \to H^{i+1}(\mathcal{F}''(m-i)) \to \ldots.$$

The first term is vanishes since $\mathcal{F}$ is m-regular and the last term vanishes since $\mathcal{F}''$ is $(m+1)$-regular so the middle term vanishes. The final conclusion follows from the proposition.

□

Now that we have the language of CM regularity, we can state the uniform CM regularity theorem in its usual form and sketch the proof.

Theorem 1. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf $\mathcal{F} \subset \mathcal{O}^{\oplus m}_{\mathbb{P}^n_k}$ with Hilbert polynomial P, $\mathcal{F}$ is N-regular.*

Proof. We will induct on n. Let H be a general hypreplane section as before and consider the sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0.$$

Now $\mathcal{F}_H \subset \mathcal{O}^{\oplus m}_H$ and the Hilbert polynomial of $\mathcal{F}_H$ is given by $P(d) - P(d-1)$ which depends only on P so by induction, there exists an N_1 depending only on P, m, and n such that $\mathcal{F}_H$ is N_1-regular.

Now consider the long exact sequence

$$\ldots \to H^{i-1}(\mathcal{F}_H(d+1)) \to H^i(\mathcal{F}(d)) \to H^i(\mathcal{F}(d+1)) \to H^i(\mathcal{F}_H(d+1)) \to \ldots.$$

For all $i \geq 2$ and $d \geq N_1 - i$, the terms with H vanish by conclusion (1) of the proposition. Thus $H^i(\mathcal{F}(d)) \to H^i(\mathcal{F}(d+1))$ is an isomorphism in this range. By Serre vanishing, these cohomology groups also vanish for d large enough so we get that $H^i(\mathcal{F}(d)) = 0$ for $i \geq 2$ and $d \geq N_1 - i$.

We need to control the groups $H^1(\mathcal{F}(d))$. Consider the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} = \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m} \to \mathcal{Q} \to 0.$$

Then $\mathcal{Q}$ has Hilbert polynomial $P'(d) = m\binom{n+d}{d} - P(d)$. By the long exact sequence of cohomology and the fact that $H^i(\mathcal{E}(a)) = 0$ for all $i > 0$ and $a > 0$, the vanishing of $H^i(\mathcal{F}(d))$ for $i \geq 2$ and $d \geq N_1 - i$ implies the vanishing $H^i(\mathcal{Q}(d)) = 0$ for all $i \geq 1$ and $d \geq N_1 - i$. In particular $\mathcal{Q}$ is N_1-regular. Then $H^0(\mathcal{Q}(d))$ surjects onto $H^1(\mathcal{F}(d))$ and has rank given by $P'(d)$ for all $d \geq N_1 - 1$. Thus

$$\dim H^1(\mathcal{F}(d)) \leq P'(d)$$

so we have uniform control on $H^1(\mathcal{F}(d))$. We conclude by the following lemma.

Lemma 1. *The sequence* $\{\dim H^1(\mathcal{F}(d))\}$ *for* $d \geq N_1 - 1$ *is strictly decreasing to* 0.

Proof. Consider the long exact sequence associated to

$$0 \to \mathcal{F}(d) \to \mathcal{F}(d+1) \to \mathcal{F}_H(d+1) \to 0.$$

Since $\mathcal{F}_H$ is N_1-regular, we have $H^1(\mathcal{F}_H(d)) = 0$ for all $d \geq N_1 - 1$ and so $H^1(\mathcal{F}(d)) \to H^1(\mathcal{F}(d+1))$ is surjective. Thus the sequence is weakly decreasing. Suppose that for some d_0, $H^1(\mathcal{F}(d_0)) \cong H^1(\mathcal{F}(d_0+1))$. The previous map

$$\varphi_{d_0} : H^0(\mathcal{F}(d_0+1)) \to H^0(\mathcal{F}_H(d_0+1))$$

is surjective. Since $\mathcal{F}_H$ is N_1-regular, then the map

$$H^0(\mathcal{F}_H(d_0+1)) \otimes H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{F}_H(d_0+2))$$

is surjective and by commutativity of the diagram

$$\begin{array}{ccc} H^0(\mathcal{F}(d_0+1)) \otimes H^0(\mathcal{O}(1)) & \longrightarrow & H^0(\mathcal{F}_H(d_0+1)) \otimes H^0(\mathcal{O}_H(1)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{F}(d_0+2)) & \xrightarrow{\varphi_{d_0+1}} & H^0(\mathcal{F}_H(d_0+2)) \end{array}$$

we conclude that φ_{d_0+1} is surjective. Thus $H^1(\mathcal{F}(d_0+1)) \cong H^1(\mathcal{F}(d_0+2))$ by the long exact sequence and so on. It follows that if $\dim H^1(\mathcal{F}(d_0)) = \dim H^1(\mathcal{F}(d_0+1))$ for some d_0, then $\dim H^1(\mathcal{F}(d_0)) = H^1(\mathcal{F}(d))$ for all $d \geq d_0$. On the other hand, this vanishes for $d \gg 0$ and so $H^1(\mathcal{F}(d_0)) = 0$. Thus the sequence must strictly decrease until it hits zero. □

Now by the monotonicity of the sequence above we see that if $N_2 := \dim H^1(\mathcal{F}(N_1 - 1))$, then $H^1(\mathcal{F}(d)) = 0$ for all $d \geq N_1 - 1 + N_2$. Now $N_2 \leq P'(N_1 - 1)$ by the previous discussion and so $H^1(\mathcal{F}(d)) = 0$ for all $d \geq N_1 - 1 + P'(N_1 - 1)$ and so $\mathcal{F}$ is $N_1 - P'(N_1 - 1)$ regular. This quantity depends only on P, m and n and so we are done. □

2 Flattening stratifications

We will now address the existence of flattening stratifications. Recall the statement.

Theorem 2. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \bigsqcup_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

Let us first consider the special case where f is the identity map $S \to S$ so that $\mathcal{F}$ is a coherent sheaf on S. Then $\mathcal{F}$ is flat if and only if it is locally free and the Hilbert polynomial of the fiber $\mathcal{F}_s$ is simply its dimension $\dim_{k(s)} \mathcal{F}_s$ over the residue field.

Proposition 2. *Let $\mathcal{F}$ be a coherent sheaf on S Noetherian. Then there exists a finite locally closed stratification $\{S_d\}$ of S such that $\mathcal{F}|_{S_d}$ is locally free of rank d. Moreover, for any locally Noetherian scheme T, a morphism $\varphi : T \to S$ factors as $T \to S_d \subset S$ if and only if $\varphi^*\mathcal{F}$ is locally free of rank d.*

Proof. First, note that by the universal property of the strata S_d, they are unique. In particular, if $I \subset S$ is an open subset, then the stratum U_d for $\mathcal{F}|_U$ is the pullback of S_d, if it exists, to U. Thus, if we prove the proposition for an open affine cover of S, it will follow for S. Therefore, without loss of generality, we may replace S by an affine open $\operatorname{Spec} A \subset S$ and suppose that $\mathcal{F}$ is the coherent sheaf associated to a finitely generated module M.

Let $s \in S$ and suppose that the rank of the fiber $\mathcal{F}_s = M \otimes k(s)$ is d. By Nakayama's lemma, we may lift the d generators of $M \otimes k(s)$ to d sections $A^{\oplus d} \to M$ which, after shrinking to a smaller open subset of Spec A, we may suppose is a surjective map. Thus we get a resolution

$$A^{\oplus e} \to A^{\oplus d} \to M \to 0.$$

By construction, the last map is an isomorphism after tensoring with $k(s)$, thus we have $\psi_{ij}(s) = 0$ for all (i, j), where the first map is given by the matrix (ψ_{ij}). Now M is locally free if and only if it has constant fiber dimension d if and only if the functions ψ_{ij} vanish, $\psi_{ij} = 0$ for all (i, j). Thus we can consider the subscheme $S_d \subset S$ given by the vanishing of all these ψ_{ij}. It is a closed subscheme containing $s \in S$.

By right exactness of pullbacks, for any $\varphi : T \to S$, pullback of the above resolution gives us a resolution

$$\mathcal{O}_T^{\oplus e} \xrightarrow{(\varphi^*\psi_{ij})} \mathcal{O}_T^{\oplus d} \longrightarrow \varphi^*\mathcal{F} \longrightarrow 0.$$

It is clear that $\varphi^*\psi_{ij}(t) = 0$ if and only if $\psi_{ij}(s) = 0$ where $s = \varphi(t)$. On the other hand, $\varphi^*\mathcal{F}$ is locally free of rank d if and only if $\varphi^*\psi_{ij} = 0$ if and only if φ factors through S_d.

By construction, each $s \in S$ is in some stratum S_d, namely for $d = \dim M \otimes k(s)$. Finally, by Noetherian induction, the locally closed stratification $\{S_d\}$ is finite since the set of ranks of fibers of the coherent sheaf $\mathcal{F}$ on the noetherian S is finite.

□