Problem set 1

Due 10/7/2019

1. If you haven't done so before, prove the Yoneda lemma. That is, if $\mathcal{C}$ is a category, X is an object of $\mathcal{C}$ and $F : \mathcal{C}^{op} \to Set$ is a presheaf, show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(h_X, F) \cong F(X).$$

   Conclude that the functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$ given by $X \mapsto h_X$ is fully faithful.

2. Are the following functors representable? If so find the representing object and universal family and if not show why not.

   - The functor $GL_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to $GL_n(\mathcal{O}_T(T))$;
   - The functor $Nil_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to the set of elements $f \in \mathcal{O}_T(T)$ with $f^n = 0$.
   - The functor $Nil : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to the set of all nilpotent elements $f \in \mathcal{O}_T(T)$.
   - The functor $F_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking T to the set $(f_1, \ldots, f_n) \in \mathcal{O}_T(T)^{\oplus n}$ such that for each $t \in T$, there exists an i with $f_i(t) \neq 0$.
   - The functor $F_n/\mathbb{G}_m : Sch_{\mathbb{Z}}^{op} \to Set$ taking T to the set $F_n(T)/\sim$ where

$$(f_1, \ldots, f_n) \sim (f'_1, \ldots, f'_n)$$

     if and only if there exists a unit $u \in \mathcal{O}_T(T)^\times$ with $f'_i = uf_i$ for all i.

3. In the last example above, what happens if we sheafify the functor $F_n/\mathbb{G}_m$?

4. Prove the following statement we used in class. Let $f : X \to Y$ be a morphism locally of finite type between locally Noetherian schemes. Show that f is a closed embedding if and only if it is a proper monomorphism. Recall that a morphism is a monomomorphism if for all T, the induced map $\mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(T, Y)$ is injective. (Hint: you might need to use the famous result that a morphism is finite if and only if it is proper and has finite fibers.)

5. In this exercise, we will prove the generic freeness theorem.

   Theorem 1. *Let A be a Noetherian integral domain and B a finitely generated A-algebra. For any finite B-module M, there exists an $f \in A$ such that M_f is a free A_f-module.*

To make the argument easier, let us say that an A-algebra B satisfies generic freeness if for any finite B-module M, there exists $f \in A$ such that M_f is a free A_f module. Then we want to show that any finitely generated A-algebra satisfies generic freeness.

(a) Show that it suffices to prove that the polynomial algebra $A[t_1, \ldots, t_n]$ satisfies generic freeness.

(b) Show that A itself satisfies generic freeness.

(c) Let B be an A-algebra and let M be a finitely generated $B[t]$-module. Let $m_1, \ldots, m_k$ be a finite set of $B[t]$-module generators. We will define a filtration of M by finite B-modules as follows. Let M_0 be the B-module generated by $m_1, \ldots, m_k$ and let $M_{n+1} = M_n + tM_n$. Note that each M_n is a finite B-module. Show that for each $n \gg 0$, multiplication by t is an isomorphism $M_n/M_{n-1} \to M_{n+1}/M_n$. Hint: consider the graded $B[t]$ module

$$\bigoplus_{n \geq 0} M_n/M_{n-1}.$$

(d) Let M be an A-module with a filtration $M_0 \subset M_1 \subset \ldots M_n \subset \ldots M$. Suppose that M_{n+1}/M_n is a free A-module for each n. Show that M is a free A-module.

(e) Suppose that B is an A-algebra which satisfies generic freeness. Use parts (c) and (d) above to conclude that $B[t]$ satisfies generic freeness.

(f) Conclude that any finitely generated A-algebra B satisfies generic freeness.