Problem set 2

Due 11/06/2019

1. Let R be a Noetherian ring, $I \subset R$ a proper ideal, M and R-module such that M/IM is flat over R/I. Suppose $\mathrm{Tor}_1^R(R/I, M) = 0$. Show the following holds for any $n \geq 1$:

   (a) $M/I^n M$ is a flat R/I^n module, and

   (b) for any R-module N that is annihilated by I^n, we have

   $$\mathrm{Tor}_1^R(N, M) = 0.$$

   Hint: for (a) use the fact that an R-module M is flat if and only for every ideal $J \subset R$,

   $$J \otimes_R M \to M$$

   is injective.

2. Let S be a Noetherian scheme, $U \subset S$ an open subscheme, and $\mathcal{E}$ a coherent sheaf on U. Show that there exists a coherent sheaf $\mathcal{E}'$ on S such that $\mathcal{E}'|_U = \mathcal{E}$.

3. Let $p : Y \to S$ be a proper morphis, $Z \subset Y$ a closed subscheme, and $\mathcal{F}$ a coherent sheaf on Y. Show that there exists an open subscheme $U \subset S$ such that a morphism $\varphi : T \to S$ factors through U if and only if the support of $\mathcal{F}_T$ on Y_T is disjoint from Z_T.

4. Recall the existence theorem on the Grothendieck complex.

   Theorem 1. *Let $f : X \to S$ be a proper morphism over a Noetherian affine scheme $S = \mathrm{Spec}\, A$ and $\mathcal{F}$ a coherent sheaf on X flat over S. Then there exists a finite complex*

   $$K^\bullet = (0 \to K^0 \to K^1 \ldots \to K^m \to 0)$$

   of finitely generated projective A-modules such that for all A-modules M, there are functorial isomorphism

   $$H^p(X, \mathcal{F} \otimes_A M) \cong H^p(K^\bullet \otimes_A M).$$

   Let $f : X \to S$ be a proper morphism over a Noetherian scheme S and let $\mathcal{E}, \mathcal{F}$ be coherent sheaves on X with $\mathcal{F}$ flat over S. Suppose that $\mathcal{E}$ admits a locally free resolution. The goal of this exercise is to use the Grothendieck complex to show that the functor $Sch_S \to Set$ given by

   $$T \mapsto \mathrm{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$$

   is representable by a scheme over S.

In fact we will show more. Given coherent sheaf $\mathcal{Q}$ on S, we may form the scheme

$$\mathbb{V}(\mathcal{Q}) := \mathrm{Spec}_S \mathrm{Sym}^* \mathcal{Q}$$

where $\mathrm{Sym}^* \mathcal{Q}$ is the symmetric algebra of $\mathcal{Q}$. When $\mathcal{Q}$ is locally free, $\mathbb{V}(\mathcal{Q})$ is what one might call the total space or geometric vector bundle associated to $\mathcal{Q}$. By the definition of relative Spec, the universal property of the symmetric algebra, and the push-pull adjunction, $\mathbb{V}(\mathcal{Q})$ is characterized by the universal property that

$$\mathrm{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = \mathrm{Hom}_{\mathcal{O}_T}(\varphi^* \mathcal{Q}, \mathcal{O}_T) = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{Q}, \varphi_* \mathcal{O}_T)$$

for any S-scheme $\varphi : T \to S$.

(a) Suppose $S = \mathrm{Spec}\, A$ is affine. Show that for any coherent $\mathcal{F}$ flat over S, there exists an S-module Q as well as a functorial isomorphism

$$\theta_M : f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* M) \to \mathcal{H}om_A(Q, M)$$

for any quasi-coherent A-module M.

(b) Suppose S is an arbitrary Noetherian scheme and $\mathcal{F}$ as above. Show that there exists a coherent sheaf $\mathcal{Q}$ on S and a funtorial isomorphism

$$\theta_{\mathcal{G}} : f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) \to \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

for all quasi-coherent sheaves $\mathcal{G}$ on S.

(c) Conclude from the previous proposition that for any coherent $\mathcal{F}$ which is flat over S, there exists a coherent $\mathcal{Q}$ on S such that

$$\mathrm{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = H^0(X_T, \mathcal{G}_T)$$

for each S-scheme T. Hint: you might need to use the projection formula for coherent sheaves.

(d) Suppose $\mathcal{F}, \mathcal{E}$ are coherent sheaves on X with $\mathcal{E}$ locally free and $\mathcal{F}$ flat over S. Use part (c) to show that there exists a coherent $\mathcal{Q}$ on S such that

$$\mathrm{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{E}_T, \mathcal{F}_T).$$

(e) Extend (d) to the case where $\mathcal{E}$ has a locally free resolution.