# Math 260X: Rationality Questions in Algebraic Geometry

Dori Bejleri

April 6, 2022

# Contents

# 1 Introduction and overview

## 1.1 Basic definitions and first examples

The study of rationality is one of the most classical in algebraic geometry. Indeed given a collection of polynomial equations with coefficients in a field $k$

$$\{f_1 = \ldots, f_n = 0\}$$

one of the first questions one can ask is whether the set of solutions $X(k)$ can be parametrized by rational functions.

**Definition 1.1.** Let $X$ and $Y$ be varieties defined over a field $k$.

1. A *rational* map $f\colon X \dashrightarrow Y$ is an equivalence class of pairs $(U, f)$ where $U \subset X$ is a dense open subset and $f\colon U \to Y$ is a $k$-morphism. Two pairs $(U, f)$ and $(V, g)$ are equivalent if $f|_{U\cap V} = g|_{U\cap V}$.

2. A rational map $f\colon X \dashrightarrow Y$ is said to be *birational* if there exists a rational map $g : Y \dashrightarrow X$ such that $g \circ f$ and $f \circ g$ are the identity where defined.

3. $X$ and $Y$ are said to be *birational* if there exists a birational map $f\colon X \dashrightarrow Y$.

The natural geometric reformulation of being able to parametrize $X(k)$ by rational functions is to give a rational parametrization of the variety $X$ by projective space.

**Definition 1.2.** We say that $X$ is *rational* if it is birational to projective space $\mathbb{P}^n$ where $n = \dim X$. We call a birational map $f\colon \mathbb{P}^n \dashrightarrow X$ a rational parametrization of $X$.

The following example shows that the rationality of a variety $X/k$ depends on the arithmetic of the field $k$. For this reason we will often, but not always, work with algebraically closed fields $k = \bar{k}$. However, many interesting examples come from non-closed fields.

**Example 1.3.** Consider the plane curve $C = \{x^2 + y^2 = pz^2\} \subset \mathbb{P}^2$ defined over $\mathbb{Q}$ where $p$ is a prime number.

1. When $p = 1$, we can write down an explicit rational parametrization $f\colon \mathbb{P}^1 \dashrightarrow C$

$$f(t) = \left[t^2 - 1\colon 2t\colon t^2 + 1\right].$$

This parametrization is given by the inverse of stereographic projection from the point $[0\colon 1\colon 0]$.

2. When $p \equiv 1 \mod 4$, we can write $p = a^2 + b^2$ and so we have a $\mathbb{Q}$-point given by $[a\colon b\colon 1]$ and again using projection from this point gives a birational map $C \dashrightarrow \mathbb{P}^1$.

3. When $p \equiv -1 \mod 4$, then in fact $C(\mathbb{Q}) = \emptyset$ so is not rational over $\mathbb{Q}$ but it becomes rational over the field extension $k = \mathbb{Q}(\sqrt{p})$. For the latter claim, just note that there is a $k$-point $[0\colon \sqrt{p}\colon 1]$ and projection from this point gives us a birational map $C \dashrightarrow \mathbb{P}^1$. For the former claim, suppose we have a $\mathbb{Q}$-point of $C$. By clearing denominators and common factors we may suppose that $x, y$ and $z$ are coprime integers. If $x$ and $y$ are not divisible by $p$, then $x^2 = -y^2 \mod p$ so $u^2 = -1 \mod p$ has a solution which contradicts that $p \equiv -1 \mod 4$. Otherwise $x$ and $y$ must both be divisible by $p$ but then $pz^2$ is divisble by $p^2$, contradicting that $x, y$ and $z$ are coprime.

*Remark* 1.4. Note that if $f : X \dashrightarrow Y$ is a birational map of $k$-varieties, then for any field extension $K/k$, there is a natural birational map $f_K : X_K \dashrightarrow Y_K$ defined by taking the basechange of a representative $U \to Y$ of $f$. Here we have used that if $U \subset X$ is dense then so is $U_K \subset X_K$ by flatness of $K/k$. Therefore rationality is preserved under extending the base field. In particular, rational varieties are always geometrically integral.

**Example 1.5.** We will see that the cubic surface $\{x^3 + y^3 + z^3 + w^3 = 0\} \subset \mathbb{P}^3$ is rational over $\mathbb{Q}$. An explicit rational parametrization over $\mathbb{Q}$ was written down by Elkies [21]. More generally, we will see that any smooth cubic surface is rational over an algebraically closed field of characteristic zero. However, it is very challenging to exhibit an explicit rational parametrization in general, even over $\mathbb{C}$, so we need to develop more geometric methods to prove this. For a general cubic surface $X$ over $\mathbb{Q}$, the situation becomes more interesting as now again the rationality of a cubic surface will depend on subtle arithmetic invariants, namely the Galois cohomology $H^1(G_{\mathbb{Q}}, \mathrm{Pic}(X_{\bar{Q}}))$.

## 1.2 Nearly rational varieties

There are several natural generalizations of the notion of rationality which are each interesting in their own rights and in relation to each other. The most natural one answers the question "what if we don't require our rational parametrizations to be one-to-one?"

**Definition 1.6.** A variety $X/k$ is *unirational* if there is a dominant rational map $f : \mathbb{P}^N \dashrightarrow X$. In this case, we call $f$ a *unirational parametrization.*

**Proposition 1.7.** *If $X/k$ is unirational, then there exists a unirational parametrization $f : \mathbb{P}^n \dashrightarrow X$ where $n = \dim X$.*

*Proof.* We give a proof when $k$ is an infinite field. The general case can be handled by an elementary but nontrivial algebraic argument (see for example [41, Proposition 1.1]. Suppose $f : \mathbb{P}^N \dashrightarrow X$ is a unirational parametrization defined on an open set $U$, where necessarily $N \geq n$. If $N = n$ we are done so suppose that $N > n$. Let $x \in X$ be a general point in the image of $f$ with $\dim f^{-1}(x) = N - n$. Since $k$ is infinite, there exists a hyperplane $H$ intersecting $U$ such that $\dim H \cap f^{-1}(x) < \dim f^{-1}(x)$. Then $f|_H : H \dashrightarrow X$ is a well defined rational map. Moreover, we claim that $f|_H$ is dominant. If not, by semi-continuity of fiber dimension, every nonempty fiber of $f|_H$ would have dimension $\geq (N-1) - (n-1) = N - n$, contradicting that $\dim H \cap f^{-1}(x) < N - n$. $\square$

An intermediate notion between rationality and unirationality that is often easier to work with is stable rationality.

**Definition 1.8.** We say that $X$ is *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m$. More generally, we say that $X$ and $Y$ are *stably birational* if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for some $n, m$.

*Remark* 1.9. Note that two varieties $X$ and $Y$ are birational if and only if the function fields $k(X)$ and $k(Y)$ are isomorphic as extensions of $k$. Moreover, a dominant rational map $Y \dashrightarrow X$ is equivalent to a field extension $k \subset k(X) \subset k(Y)$. Thus rationality (resp. stable rationality, resp. unirationality) is equivalent to the purely field theoretic statement that $k(X)$ is purely transcendental over $k$ (resp. $k(X)(t_1, \ldots, t_m)$ is purely transcendental over $k$, resp. $k(X)$ is a subfield of a purely transcendental extension $k(t_1, \ldots, t_N)$). In particular, rationality (resp. stable rationality, resp. unirationality) depends only on the birational equivalence class of $X$. For many arguments, it is necessary to pick a projective model for $X$. Moreover, when $\mathrm{char}(k) = 0$, we can (and often will) assume $X$ is also smooth thanks to Hironaka's theorem.

The following result due to Lang and Nishimura guarantees that $X(k) \neq \emptyset$ for proper unirational varieties.

**Lemma 1.10.** *Suppose $f : X \dashrightarrow Y$ is a rational map with $Y$ proper and $X$ smooth. If $X$ has a $k$-rational point then so does $Y$.*

*Proof.* We will induct on the dimension $n$ of $X$. If $n = 0$, then $f$ is a morphism so the image of the rational point on $X$ gives a rational point on $Y$. In general, let $x \in X(k)$ be a raitonal point and consider the blowup $X' = \mathrm{Bl}_x X$. By the valuative criterion for properness, the composition $X' \to X \dashrightarrow Y$ extends across codimension 1 points of $X'$. In particular, the restriction to the exceptional divisor $E \cong \mathbb{P}^{n-1}$ is a well defined rational map $\mathbb{P}^{n-1} \dashrightarrow Y$ so by induction, $Y$ has a rational point.

□

*Remark* 1.11. Neither assumptions in the above lemma can be dropped. Indeed $Y = \bar{Y} \setminus \bar{Y}(k)$ for a variety $\bar{Y}$ with finitely many $k$-points gives a counterexample when $Y$ is not proper. When $X$ is not smooth, then the exceptional divisor of $X' \to X$ need not be isomorphic to projective space over $k$, consider for example the affine cone $x^2 + y^2 + z^2 = 0$ in $\mathbb{A}^3$ over $\mathbb{Q}$ with unique rational point $(0, 0, 0)$.

Finally, we introduce the notion of rationally connected which we will see is especially well-behaved over an algebraically closed field of characteristic 0.

**Definition 1.12.** Let $k = \bar{k}$ be an algebraically closed field of characteristic 0 and let $X/k$ be a smooth projective variety. We say that $X$ is *rationally connected* if for any two general points $x, y \in X$, there exists a map $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$.

We have the following simple but important implications.

$$\text{rational} \implies \text{stably rational} \implies \text{unirational} \implies \text{rationally connected} \tag{1}$$

The first two are by definition. For the third, let $f : \mathbb{P}^N \dashrightarrow X$ be a unirational parametrization. Then a general point of $X$ is in the image of $f$ and for any two points $x, y$ in the image of $f$, we can pick preimages $x', y' \in \mathbb{P}^N$ and let $L \subset \mathbb{P}^N$ be the line through $x'$ and $y'$. Then $f|_L$ gives the required rational curve connecting $x$ and $y$.

One of the major motivating questions in the field is to what extent are any of these implications reversible? For the remainder of this section we will give an overview of the history of this problem and some of the topics we plan to cover over the semester.

## 1.3 The Lüroth problem

One of the first theorems in the subject concerns the reversibility of the implication rational $\implies$ unirational in the case of dimension 1.

**Theorem 1.13** (Lüroth [36])**.** *Suppose $X$ is a 1-dimensional unirational variety over $k$. Then $X$ is rational over $k$.*

*Proof.* We give a geometric argument which works when $\mathrm{char}(k) = 0$. The general case can be proved by an elementary but involved field theoretic argument (see [41, Theorem 1.3]).

By taking a projective closure and normalizing, we may assume that $X$ is smooth and projective. Then our rational parametrization extends to a surjective morphism $f\colon \mathbb{P}^1 \to X$. Since $f$ is separable, we have the Riemann–Hurwitz formula which in this case reads

$$-2 = \deg(f)(2g-2) + \deg(R)$$

where $g$ is the genus of $X$ and $R$, the ramification divisor, is effective. Then $g < 1$ so $g = 0$. By Lang–Nishimura 1.10, $X$ has a rational point. Now we conclude by the following lemma.

**Lemma 1.14.** *If $X/k$ is a smooth projective curve of genus 0 with a rational point, then $X \cong \mathbb{P}^1$.*

*Proof.* Let $p \in X(k)$ be a $k$-point. The line bundle $\mathscr{O}_X(p)$ has degree 1 and by Serre duality, $H^1(X, \mathscr{O}_X(p)) = 0$ since $X$ has genus 0. Then by Riemann–Roch, we have

$$h^0(X, \mathscr{O}_X(p)) = 2.$$

Thus, the complete linear series $H^0(X, \mathscr{O}_X(p))$ induces a morphism $X \to \mathbb{P}^1$ with degree 1 fibers. Such a morphism is necessarily an isomorphism so $X$ is rational.

□

□

So in dimension one, rational and unirational are equivalent. Thus it is natural to wonder what happens in higher dimensions.

**Question 1.15** (Lüroth Problem)**.** Under what circumstances is a unirational variety rational? Equivalently, when is a subfield $k \subset K \subset k(t_1, \ldots, t_n)$ purely transcendental over $k$?

More generally, we sometimes refer to the question of invertibility of any of the implications in Equation (1) as the Lüroth Problem.

In dimension 2 over an algebraically closed field of characteristic 0, the Lüroth problem is solved by the following famous Theorem of Castelnuovo which we will prove later in the course.

**Theorem 1.16** (Castelnuovo's Theorem)**.** *Suppose that $k = \bar{k}$ and $\mathrm{char}(k) = 0$. If $X/k$ is a smooth projective surface with*

$$h^1(X, \mathscr{O}_X) = h^0(X, \omega_X^{\otimes 2}) = 0,$$

*then $X$ is rational.*

*Remark* 1.17. Here $\omega_X = \det \Omega^1_X$ is the canonical bundle of the smooth projective variety $X$.

Thus in dimension $\leq 2$ and over an algebraically closed field of characteristic 0, rationality is completely captured by numerical invariants, namely the genus in dimension 1 and the invariants $h^1(X, \mathscr{O}_X) = h^0(X, \omega_X^{\otimes 2})$ in dimension 2. We will see that Castelnuovo's Theorem implies the following generalization of Lüroth's Theorem.

**Corollary 1.18.** *Suppose that $k = \bar{k}$ and $\mathrm{char}(k) = 0$. If $X/k$ is a smooth projective unirational surface, then $X$ is rational.*

*Remark* 1.19. We will see that there are interesting arithmetic examples of unirational but not rational surfaces over a non-closed field of characteristic 0. Moreover, Castelnuovo's Theorem is false in characteristic $p$.

In dimension $\geq 3$, the Lüroth Problem has a negative answer and and there is a long history of interesting examples and counterexamples to the reverse of the implications in Equation (1) which we will discuss later in the course.

### 1.3.1 Clemens–Griffiths

Let $X$ be a smooth projective threefold over the complex numbers and suppose that $h^{1,0}(X) = h^{3,0}(X) = 0$. Clemens and Griffiths [13] associated to $X$ a principally polarized Abelian variety (ppav)

$$\mathrm{IJ}(X) := \frac{H^{2,1}(X,\mathbb{C})^\vee}{H_3(X,\mathbb{Z})},$$

the *intermediate Jacobian* of $X$. They show that if $X$ is rational, then

$$\mathrm{IJ}(X) \cong \mathrm{Jac}(C)$$

as ppavs where $C$ is a possibly disconnected smooth projective curve.

Specializing to the case of a smooth cubic threefold $X \subset \mathbb{P}^4$, they show that for such $X$, $\mathrm{IJ}(X)$ is never isomorphic to $\mathrm{Jac}(C)$. On the other hand, a geometric construction via conic bundles shows that such $X$ is unirational. Thus we obtain examples of unirational but not rational varieties.

**Theorem 1.20** (Clemens–Griffiths [13])**.** *Smooth cubic threefolds over $\mathbb{C}$ are unirational but not rational.*

**Question 1.21.** Is a smooth cubic threefold stably rational?

### 1.3.2 Artin–Mumford

Artin and Mumford [1] introduced an invariant which can be used to detect stable rationality. This invariant is the torsion subgroup of the 3rd integral homology group,

$$H_3(X,\mathbb{Z})_{tors}$$

which is closely related to the Brauer group $\mathrm{Br}(X)$. Artin and Mumford showed that $H_3(X,\mathbb{Z})_{tors}$ is a stable birational invariant so that in particular, $H_3(X,\mathbb{Z})_{tors} = 0$ if $X$ is stably rational. Using this invariant, they constructed examples of unirational but not stably rational varieties.

**Theorem 1.22** (Artin–Mumford [1])**.** *There exists a variety $X/\mathbb{C}$ which is unirational but not stably rational.*

### 1.3.3 Iskovskikh–Manin

The approach of Iskovskikh and Manin [27] is based on the following observation.

**Definition 1.23.** A *birational automorphism* of $X$ is a birational map $X \dashrightarrow X$. The group of all birational automorphisms will be denoted $\mathrm{Bir}(X)$.

*Observation* 1.24. If $X$ and $Y$ are birational then

$$\mathrm{Bir}(X) \cong \mathrm{Bir}(Y).$$

In particular, rational varieties must have very large automorphism group since $\mathrm{Bir}(\mathbb{P}^n) =: \mathrm{Cr}_n$ also known as the *Cremona group*, is very large and in particular infinite.

**Example 1.25.** If $k$ is an infinite field, then $\mathrm{PGL}_{n+1}(k) \subset \mathrm{Cr}_n$ shows that $\mathrm{Cr}_n$ is infinite. For $n \geq 2$, $\mathrm{Cr}_n$ is still infinite even over finite fields. For example there exist infinitely automorphisms of $\mathbb{A}^n$ of the form

$$(x_1, \ldots, x_n) \mapsto (x_1 + P(x_2, \ldots, x_n), x_2, \ldots, x_n)$$

where $P \in k[x_2, \ldots, x_n]$ is any polynomial. In fact $\mathrm{Cr}_n$ is not finitely generated [10, Proposition 3.6]

**Theorem 1.26.** *(Iskovskikh–Manin [27]) Let $X \subset \mathbb{P}^4$ be a smooth quartic threefold. Then* $\mathrm{Bir}(X)$ *is finite. In particular, $X$ is not rational.*

On the other hand, B. Segre gave the following example of a quartic threefold which is unirational, yielding another counterexample to the Lüroth Problem.

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0$$

### 1.3.4 Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer

Using arithmetic techniques, Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer [4] constructed examples of stably rational but not rational surfaces over a non-closed function field $K/\mathbb{C}$. By spreading out they then obtain higher dimensional examples over $\mathbb{C}$.

**Theorem 1.27** (Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer [4])**.** *There exists an irrational threefold $X/\mathbb{C}$ such that $X \times \mathbb{P}^3$ is rational.*

### 1.3.5 Unirational versus rationally connected

The above examples which we will cover in detail in this course show that the first two arrows in Equation (1) are strict. The last one is also expected to be strict but this is still an open problem.

$$\text{rational} \xRightarrow{\text{strict}} \text{stably rational} \xRightarrow{\text{strict}} \text{unirational} \xRightarrow{\text{strict?}} \text{rationally connected} \tag{2}$$

**Question 1.28.** Do there exist smooth rationally connected varieties over $\mathbb{C}$ which are not unirational?

## 1.4 Rationality in families

In the latter part of the course we will discuss the behavior of rationality in families. Given a family $f : X \to S$ of varieties over $k$, it is natural to study the loci of points $s \in S$ such that $X_s$ is rational (resp. stably rational, resp. unirational). When is rationality deformation invariant in families? Do there exist families with both rational and irrational fibers? Can we use specialization methods to study the rationality of general members of our family?

We begin with the following basic observations.

*Observation* 1.29. When $k = \mathbb{C}$, rationality is both an open and closed condition in a smooth projective family $f : X \to S$ of relative dimension $\leq 2$. This follows from the numerical characterization of rationality in small dimensions.

*Observation* 1.30. Rationality does not behave well when we degenerate to singular varieties.

**Example 1.31.**

1. Suppose $f : X \to \operatorname{Spec} R$ is a family of elliptic curves over a DVR degenerating to a nodal cubic curve. Then the generic fiber is not rational but the special fiber is rational.

2. Suppose $f : X \to \operatorname{Spec} R$ is a family of smooth cubic hypersurfaces over a DVR degenerating to the cone over an elliptic curve. Then the general fiber of $f$ is rational but the special fiber is not rational.

*Remark* 1.32. We will see that the property of being rationally connected is both open and closed in smooth projective families of complex varieties.

### 1.4.1 The specialization method for stable irrationlity

Much of the recent progress on (stable) rationality has been spurred on by a specialization method, first introduced by Voisin and generalized by Colliot-Thélène–Pirutka, for proving that very general members of a family are not stably rational. The obstruction to stable rationality used in this method comes from studying algebraic cycles on $X$. Given a projective family $f : X \to S$, the upshot of this method is that the existence of a singular fiber with mild singularities and nonzero obstruction implies that the very general fiber of $f$ is not stably rational. We will state the precise theorem here but postpone the definitions until later.

**Theorem 1.33** (Voisin, Colliot-Thélène–Pirutka [46, 14])**.** *Let $k = \bar{k}$ be an uncountable field. Suppose $f : X \to S$ is a flat projective family over an integral variety $S$. Suppose that there exists a point $0 \in S$ such that the fiber $X_0$ satisfies*

1. *$X_0$ admits a universally $\mathrm{CH}_0$-trivial resolution of singularities $\mu : Y \to X_0$, and*

2. *$Y$ is not universally $\mathrm{CH}_0$-trivial.*

*Then the very general fiber of $f$ is not stably rational.*

Applications of this method have led to many examples due to Hassett, Kresch, Tschinkel, Pirutka, Schreieder and others of the following surprising behavior of rationality in families.

**Theorem 1.34.** *There exist smooth projective families $f : X \to S$ of complex varieties over an integral base $S$ such that*

1. *the very general fiber of $f$ is not stably rational, and*

2. *the locus $\{s \in S(\mathbb{C}) \mid X_s \text{ is rational}\}$ is dense in the complex topology on $S(\mathbb{C})$.*

### 1.4.2 Specialization of (stable) rationality

In the examples alluded to in Theorem 1.34, the generic fiber of the family is not rational but there are many rational fibers. We can also ask what happens when the generic fiber of the family is (stably) rational.

**Theorem 1.35** (Nicaise–Shinder, Kontsevich–Tschinkel, Nicaise–Ottem [40, 34, 39])**.** *Suppose* $\mathrm{char}(k) = 0$ *and let $f : X \to S$ be a smooth projective family over a smooth connected curve $S$. Suppose the generic fiber of $f$ is (stably) rational over the function field $k(S)$. Then $X_s$ is (stably) rational for all $s \in S$.*

The proofs of this theorem involve constructing an appropriate specialization map of variants of the Grothendieck ring of varieties of the function field $k(S)$ to that of the residue field $k(s)$ and envoking the Weak Factorization theorem for birational morphisms. The analagous question for unirationality is still open.

**Question 1.36.** Does unirationality specialize in one parameter families?

# 2 Plurigenera and Castelnuovo's Theorem

A fundamental goal in the study of rationality is to produce numerical invariants which can be used to distinguish between rational and irrational varieties. We saw that for smooth projective curves over an algebraically closed field, the genus $g$ fulfills this role. The *plurigenera* are the higher dimensional generalizations of the genus of a curve.

**Definition 2.1.** Let $X/k$ be a smooth projective variety. The $m^{th}$ *plurigenus* $P_m$ is

$$P_m(X) := h^0(X, \omega_X^{\otimes m})$$

where $\omega_X := \Lambda^{\dim X}\Omega_X$ is the canonical bundle of $X$.

*Remark* 2.2. The first plurigenus $P_1 = h^0(X, \omega_X)$, sometimes denoted $p_g$, is the *geometric genus* which agrees with the genus of a curve in dimension 1.

In this section we will see that the plurigenera, and more generally the invariants

$$P_{m,n}(X) := h^0(X, (\Omega_X^n)^{\otimes m})$$

are birational invariants. Note here that $\Omega_X^n := \Lambda^n\Omega_X$ and that $P_{m,d}(X) = P_m$ where $d = \dim X$.[1] Moreover, these invariants vanish for projective space so that in particular, they give an obstruction to rationality. We will also sketch a proof of Castelnuovo's Theorem which shows that these are complete invariants for rationality of surfaces over an algebraically closed field.

**Theorem 2.3** (Castelnuovo's Theorem)**.** *Suppose that $k = \bar{k}$ and $\mathrm{char}(k) = 0$. If $X/k$ is a smooth projective surface with*

$$h^1(X, \mathscr{O}_X) = h^0(X, \omega_X^{\otimes 2}) = 0,$$

*then $X$ is rational.*

*Remark* 2.4. Under the assumptions of the theorem, classical Hodge theory shows that

$$h^1(X, \mathscr{O}_X) = h^0(X, \Omega_X) = P_{1,1}(X)$$

## 2.1 Birational invariance of plurigenera

In this section we show that $P_{m,n}(X)$ is a birational invariant for all $m, n$ and $X$ smooth and proper.

**Proposition 2.5.** *Let $X$ and $Y$ be smooth and proper $k$-varieties. If $f : X \dashrightarrow Y$ is a separable, dominant rational map, then*

$$h^0(X, (\Omega_X^n)^{\otimes m}) \geq h^0(Y, (\Omega_Y^n)^{\otimes m}) \text{ for all } n, m \geq 0.$$

*Proof.* Since $X$ is normal and $Y$ is proper, there exists a dense open subset $U \subset X$ such with $\mathrm{codim}_X(X \subset U) \geq 2$ such that $f$ restricts to a morphism $f : U \to Y$.

Since $X$ is normal, $(\Omega_X^n)^{\otimes m}$ is locally free, and the complement of $U$ has high codimension, we have that

$$h^0(X, (\Omega_X^n)^{\otimes m}) = h^0(U, (\Omega_X^n)\big|_U^{\otimes m}) = h^0(U, (\Omega_U^n)^{\otimes m}).$$

Thus it suffices to show that

$$h^0(U, (\Omega_U^n)^{\otimes m}) \geq h^0(Y, (\Omega_Y^n)^{\otimes m}).$$

[1]This notation $P_{m,n}$ is not standard.

We claim in fact that the pullback map

$$H^0(Y,(\Omega^n_Y)^{\otimes m}) \to H^0(U,(\Omega^n_U)^{\otimes m})$$

via $f : U \to Y$ is injective. Indeed it factors as

$$H^0(Y,(\Omega^n_Y)^{\otimes m}) \to H^0(U,(f^*\Omega^n_Y)^{\otimes m}) \to H^0(U,(\Omega^n_U)^{\otimes m}) \tag{3}$$

where the first map is the usual pullback of sections and the second map is induced by the natural map $df : f^*\Omega_Y \to \Omega_U$. The first map is injective since $f$ is dominant and $(\Omega^n_Y)^{\otimes m}$ is locally free. For the second map, note that $f$ is generically smooth by the seperability assumption so $df : f^*\Omega_Y \to \Omega_U$ is generically injective, but $f^*\Omega_Y$ is torsion free so $df$ is injective. Since both sheaves are vector bundles, $(\Lambda^n df)^{\otimes m}$ is also injective and thus the second map in (3) is also injective by taking global sections.

□

**Corollary 2.6** (Plurigenera are birationally invariant)**.** *If $X$ and $Y$ as above are birational, then*

$$P_{m,n}(X) = P_{m,n}(Y) \text{ for all } m,n \geq 0.$$

*Proof.* since $X$ and $Y$ are birational, there exist rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ which are inverses on a dense open subset. In particular, $f$ and $g$ are dominant and separable so applying Proposition 2.5 to both maps gives the required equality. □

In characteristic 0, every map is separable so we obtain the following.

**Corollary 2.7.** *If $\mathrm{char}(k) = 0$ and $f : X \dashrightarrow Y$ is a dominant map with $X$ and $Y$ as above, then*

$$h^0(X,(\Omega^n_X)^{\otimes m}) \geq h^0(Y,(\Omega^n_Y)^{\otimes m}) \text{ for all } n,m \geq 0.$$

**Corollary 2.8.** *If $\mathrm{char}(k) = 0$ then for all $n \geq 0$, $h^n(X,\mathscr{O}_X)$ is a birational invariant for smooth and proper $X$.*

*Proof.* By general reductions, we may assume that $k = \mathbb{C}$. Then the Hodge decomposition and Hodge symmetries tell us that

$$h^n(X,\mathscr{O}_X) = h^0(X,\Omega^n_X) = P_{1,n}$$

which is a birational invariant. □

*Remark* 2.9. Corollary 2.8 also holds over fields of characteristic $p$ but the proof is much harder and more recent due to Chatzistamatiou–Rülling [11].

**Example 2.10.** Here we compute that $P_{m,n}(\mathbb{P}^N) = 0$ for $m,n \geq 1$. The Euler sequence tells us that

$$0 \to \Omega_{\mathbb{P}^N} \to \mathscr{O}_{\mathbb{P}^N}(-1)^{\oplus N+1} \to \mathscr{O}_{\mathbb{P}^N} \to 0. \tag{4}$$

Taking exterior powers, we obtain an injection

$$0 \to \Omega^n_{\mathbb{P}^N} \to \mathscr{O}_{\mathbb{P}^N}(-1)^{\oplus\binom{N+1}{n}}. \tag{5}$$

Since $h^0(\mathbb{P}^N,\mathscr{O}_{\mathbb{P}^N}(-1)) = 0$, we conclude that $P_{1,n}(\mathbb{P}^N) = 0$. Now we induct on $m$. Tensoring the sequence (5) with $(\Omega^n_{\mathbb{P}^N})^{\otimes m-1}$ and taking global sections gives us an injection

$$H^0(\mathbb{P}^N,(\Omega^n_{\mathbb{P}^N})^{\otimes m}) \hookrightarrow H^0(\mathbb{P}^N,(\Omega^n_{\mathbb{P}^N})^{\otimes m-1}(-1)^{\oplus\binom{N+1}{n}}).$$

Now $h^0(\mathbb{P}^N,(\Omega^n_{\mathbb{P}^N})^{\otimes m-1}(-1))) \leq h^0(\mathbb{P}^N,(\Omega^n_{\mathbb{P}^N})^{\otimes m-1}) = P_{m-1,n} = 0$ by induction so we conclude that $P_{m,n} = 0$.

By Example 2.10 and the birational invariance of plurigenera, obtain that $P_{m,n}(X)$ gives an obstruction to rationality.

**Proposition 2.11.** *Let $X/k$ be a smooth and proper variety. If $X$ is rational, then $P_{m,n}(X) = 0$ for all $m, n \geq 1$. If $\mathrm{char}(k) = 0$ and $X$ is unirational, then we have the same conclusion.*

*Proof.* The first claim follows from Example 2.10 and Proposition 2.5. The second claim follows from the same example and Corollary 2.7. □

We are now ready to see our first examples of irrational varieties in higher dimensions.

**Example 2.12.** 1. Let $X_d \subset \mathbb{P}^n$ be a smooth degree $d$ hypersurface. Then by the adjunction formula,

$$\omega_{X_d} = \omega_{\mathbb{P}^N}(d)|_{X_d} = \mathscr{O}_{X_d}(d - n - 1).$$

In particular, if $d \geq n+1$, $P_m(X_d) = h^0(X_d, \mathscr{O}_{X_d}(m(d-n-1))) > 0$ so $X_d$ is not rational, and not even unirational in characteristic 0.

2. More generally, the adjunction formula can be extended to show that a complete intersection $X \subset \mathbb{P}^n$ of degree $(d_1, \ldots, d_k)$ has canonical bundle given by

$$\omega_X = \mathscr{O}_X(d_1 + \ldots + d_k - n - 1)$$

and thus $X$ is not rational (or even unirational when $\mathrm{char}(k) = 0$) for $d_1 + \ldots + d_k \geq n+1$.

*Remark* 2.13. The assertion in Proposition 2.11 about unirationality is false in characteristic $p$. For example, there exist unirational K3 surfaces $X$ in positive characteristic. These are unirational smooth projective surfaces $X$ with $\omega_X = \mathscr{O}_X$ so in particular $P_1(X) = 1$.

By Example 2.12, rational hypersurfaces in $\mathbb{P}^n$ can only exist for low degree $d \leq n$. We will see later by constructing more sophisticated invariants that there exist hypersurfaces of degree $d \leq n$ which are not rational. Thus the plurigenera are not a complete invariant for rationality.

*Remark* 2.14. We will see later that if $X$ is a rationally connected variety then $P_{m,n}(X) = 0$ for all $m, n \geq 1$. It is famously conjectured that the vanishing $P_{m,n}(X) = 0$ implies that $X$ is rationally connected. This is implied by the deep conjectures of the minimal model program but is still open.

## 2.2 Castelnuovo's Theorem

In this section we will sketch the proof of Castelnuovo's Theorem using ideas from the minimal model program for surfaces. For the rest of this subsection we assume that $k = \bar{k}$ and $\mathrm{char}(k) = 0$.

**Theorem 2.15** (Castelnuovo's Theorem)**.** *If $X/k$ is a smooth projective surface with*

$$h^1(X, \mathscr{O}_X) = h^0(X, \omega_X^{\otimes 2}) = 0,$$

*then $X$ is rational.*

*Proof.* (Sketch) Recall that a $(-1)$-curve $E \subset X$ is a rational curve $E$ with $E^2 = -1$. By Castelnuovo's Criterion for contractibility, if $E \subset X$ is a $(-1)$-curve, then there exists a projective birational morphism $f : X \to X_1$ such that

- $X$ is smooth,

- $f$ contracts $E$ to a point, and
- $f$ maps $X \setminus E$ isomorphically onto $X_1 \setminus p$.

Then we have $0 < \rho(X_1) < \rho(X)$ where $\rho(X) := \dim NS(X)_{\mathbb{R}}$ is the *Picard* number. If $X_1$ has a $(-1)$-curve, we can further contract this curve. Eventually this process must terminate so we see that $X$ is birational to a smooth projective surface $X_0$ with no $(-1)$-curves. Since the plurigenera and rationality are invariant under birational maps, we may replace $X$ with $X_0$ and assume that $X$ has no $(-1)$-curves.

Next, suppose that there exists a curve $C$ with $K_X.C < 0$. Recall that $K_X$, the canonical divisor, is characterized by $\mathscr{O}_X(K_X) = \omega_X$. Then by results from the minimal model program, there exists a surjective morphism[2]

$$\varphi : X \to Z$$

such that $\dim Z < 2$, $\varphi$ has integral fibers, $Z$ is normal and $-K_X|_F$ ample for a general fiber $F$.

Suppose $\dim Z = 1$, then $\varphi$ is a flat projective morphism to a smooth curve with integral genus 0 fibers, that is, it is a $\mathbb{P}^1$ bundle. By Tsen's theorem, $\varphi : X \to Z$ is birational to $Z \times \mathbb{P}^1$. If $Z = \mathbb{P}^1$, then $X$ is rational. Otherwise, $Z$ is higher genus and we derive a contradiction:

$$h^1(X, \mathscr{O}_X) = h^1(Z \times \mathbb{P}^1, \mathscr{O}_{Z\times\mathbb{P}^1}) = h^1(Z, \mathscr{O}_Z) > 0.$$

If $\dim Z = 0$, then $-K_X$ is ample and $\rho(X) = 1$. This is the hardest part of the proof, but the idea is to show that there exists an ample divisor $H$ such that $-K_X = 3H$ and then show that the linear series $\varphi_{|H|}$ maps $X$ isomorphically onto $\mathbb{P}^2$, which in particular is rational.

Thus, it suffices to show that there exists a curve $C$ with $K_X.C < 0$. Suppose that $K_X.C \geq 0$ for all curves $C \subset X$ and fix an ample divisor $A$.

**Lemma 2.16.** *We have $K_X^2 \geq 0$ and $A.K_X \geq 0$.*

*Proof.* Since $K_X.C \geq 0$ for all curves, $K_X + \epsilon A$ is ample for all $\epsilon > 0$ so $(K_X + \epsilon A)^2 > 0$. Taking the limit $\epsilon \to 0$ gives $K_X^2 \geq 0$. On the other hand, an ample divisor is linearly equivalent to an effective one so $A.K_X \geq 0$ by assumption. □

Now

$$\chi(\mathscr{O}_X) = h^0(X, \mathscr{O}_X) - h^1(X, \mathscr{O}_X) + h^2(X, \mathscr{O}_X) = 1 - 0 + 0 = 1$$

where the second term is 0 by assumption and the third term is zero by Serre duality and the fact that $P_2 = 0 \implies P_1 = 0$. Then

$$h^2(X, \mathscr{O}_X(-K_X)) = h^0(X, \mathscr{O}_X(2K_X)) = 0$$

so by Riemann–Roch,

$$h^0(X, \mathscr{O}_X(-K_X)) \geq \frac{1}{2}(-K_X)(-K_X - K_X) + \chi(\mathscr{O}_X) = K_X^2 + 1 \geq 1.$$

Thus, there exists an effective divisor $D \in |-K_X|$ so

$$0 < D.A = -K_X.A \leq 0$$

which is a contradiction.

□

[2]called an extremal contraction

By Corollary 2.7 and the Hodge decomposition, we obtain that the Lüroth problem has a positive solution for surfaces over an algebraically closed field of characteristic zero.

**Corollary 2.17.** *Let $X/k$ be a smooth projective unirational surface. Then $X$ is rational.*

We also have a complete understanding of the behavior of rationality in families.

**Corollary 2.18.** *Suppose $f : X \to S$ is a smooth projective family of surfaces over an integral base scheme $S$. If $X_0$ is rational then for some $0 \in S$, then $X_s$ is rational for all $s \in S$.*

*Proof.* By Castelnuovo's Theorem, we need to show that $h^1(X_s, \mathscr{O}_{X_s}) = h^0(X_s, \omega_{X_s}^{\otimes 2}) = 0$ for all $s \in S$ assuming the vanishing holds for $s = 0$. The first vanishing follows from deformation invariance of Hodge numbers and the second from deformation invariance of plurigenera.

**Proposition 2.19.** *Let $f : X \to S$ be a smooth projective family of varieties over an integral base. Then the Hodge numbers*

$$h^{p,q}(X_s) := h^q(X_s, \Omega^p_{X_s})$$

*are constant for $s \in S$.*

*Proof.* By the Hodge decomposition, we have

$$b_k(X_s) = \sum_{p+q=k} h^{p,q}(X_s).$$

By Ehresmann's Theorem, every fiber of $f$ is diffeomorphic so $b_k(X_s) = b_k$ is constant. By upper-semicontinuity, for any $s_0 \in S$, there exists a Zariski open neighborhood $s \in U \subset X$ such that $h^{p,q}(X_s) \le h^{p,q}(X_{s_0})$ for all $s \in U$ so

$$b_k = \sum_{p+q=k} h^{p,q}(X_s) \le \sum_{p+q=k} h^{p,q}(X_{s_0}) = b_k.$$

It follows that $h^{p,q}(X_s)$ is constant for all $s \in U$. Since $S$ is quasi-compact and integral, we conclude that $h^{p,q}(X_s)$ is constant for all $s \in S$. $\square$

**Theorem 2.20** (Siu's Deformation Invariance of Plurigenera)**.** *Let $f : X \to S$ be a smooth projective family over an integral base. Then the plurigenera*

$$P_m(X_s) = h^0(X_s, \omega_{X_s}^{\otimes m})$$

*are constant for $s \in S$.*

The proof of this theorem is beyond the scope of this class. Instead we will give a direct proof of the case needed for the corollary due to Iitaka. First we note that

$$q(X_s) = h^1(X, \mathscr{O}_{X_s}) = 0 \qquad P_1(X_s) = h^0(X_s, \omega_{X_s}) = h^2(X_s, \mathscr{O}_{X_s}) = 0$$

are constant by Proposition 2.19 and in particular, $\chi(\mathscr{O}_X) = 1$. Here we have used Serre duality and the fact that $P_2(X_0) = 0 \implies P_1(X_0) = 0$. Moreover, we can contract $(-1)$-curves in families and suppose without loss of generality that $X_0$ is minimal. Then $K_{X_0}^2 \ge 0$ by the classification of minimal rational surfaces and $K_{X_s}^2 \ge 0$ for all $s \in S$ by constancy of intersection numbers.

By upper semi-continuity, there exists a dense open subset $U \subset S$ where $P_2(X_s) = 0$. Suppose the complement $Z := S \backslash U$ is non-empty. Then there exists a point $1 \in Z \subset S$ such that $P_2(X_1) \neq 0$. If

$$P_{-2}(X_1) = h^0(X_1, \omega_{X_1}^{\otimes -2}) \neq 0$$

then $\omega_X^{\otimes 2} \cong \mathscr{O}_X$ but $\omega_X \ncong \mathscr{O}_X$.

**Lemma 2.21.** *Let $L$ be a nontrivial line bundle on $X$ and suppose we have a trivialization $L^{\otimes 2} \cong \mathscr{O}_X$. Then there exists a finite étale cover $\pi : Y \to X$ with $\pi^* L \cong \mathscr{O}_Y$.*

*Proof.* Let $\mu : L^{\otimes 2} \to \mathscr{O}_X$ denote the trivialization of $L^{\otimes 2}$. Them $\mathscr{A} = \mathscr{O}_X \oplus L$ is a coherent sheaf of algebras with multiplication given by $m(a + t, a' + t') = aa' + \mu(t, t') + at' + a't$ for $a, a'$ local sections of $\mathscr{O}_X$ and $t, t'$ local sections of $L$. Then we let $Y = \operatorname{Spec}_X \mathscr{A}$ be the relative Spec. Then $\pi^* L \cong \mathscr{O}_Y$ by construction and $\pi$ is finite étale since locally it can be written as $y^2 = g(x)$ where $g(x) \neq 0$. $\square$

Thus $X_s$ is not simply connected. On the other hand, $\pi_1(X_0) = 0$ for rational varieties (we will see later this is true more generally for any rationally connected variety) and $\pi_1(X_s) \cong \pi_1(X_0)$ for all $s \in S$ which is a contradiction.

Thus $P_{-2}(X_1) = 0$ so by semi-continuity, there exists an open neighborhood $1 \in V \subset S$ such that $P_{-2}(X_s) = 0$ for all $s \in V$. On the other hand, $P_3(X_s) = h^0(X_s, \omega_{X_s}^{\otimes 3}) = h^2(X_s, \omega_{X_s}^{\otimes -2}) = 0$ for all $s \in U$. Since $U \cap V \neq \emptyset$, we can find an $s \in S$ with $P_{-2}(X_s) = P_3(X_s) = 0$ but then by Riemann–Roch,

$$P_{-2}(X_s) + P_3(X_s) \geq \chi(\mathscr{O}_X) + \frac{1}{2}(-2K_{X_s})(-2K_{X_s} - K_{X_s}) = 1 + 3K_{X_s}^2 \geq 1$$

which is a contradiction.

$\square$

# 3 Hypersurfaces of low degree

In the last section we saw that smooth projective hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \geq n+1$ are never rational. In this section we will discuss rationality of some low degree hypersurfaces.

## 3.1 Quadric hypersurfaces

The study of quadric hypersurfaces, i.e. quadratic forms, is quite rich for non-closed fields. In this section we will prove that rationality of quadrics is completely determined by the existence of a rational point.

**Proposition 3.1.** *Let $X \subset \mathbb{P}^n$ be a smooth quadric. Then $X$ is rational if and only if $X(k) \neq \emptyset$. In particular, unirational quadrics are rational.*

*Remark* 3.2. Smoothness of $X$ is essential here. For example, the quadric $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^4$ has a rational point $[0\colon 0\colon 0\colon 1]$ but is not rational over $\mathbb{Q}$. However, as the proof will indicate, it suffces for $X$ to have a *smooth* $k$-point.

*Proof.* By Lemma 1.10, if $X$ is rational it has a rational point so conversely, suppose $p \in X(k)$ is a smooth rational point. Then consider the projection $\pi_p : X \dashrightarrow \mathbb{P}^{n-1}$ away from $p$. Every line through $p$ intersects $X$ in exactly one other point so $\pi_p$ is birational. Now if $X$ is unirational, then $X$ has a rational point by Lemma 1.10 so $X$ is rational. $\square$

Next we prove a theorem of Springer [45] (see also [15, Proposition 2.1]) which allows us to find rational points on quadrics.

**Theorem 3.3.** *Let $X/k$ be a quadric hypersurface and $K/k$ a finite extension of odd degree. If $X(K) \neq \emptyset$ then $X$ has a $k$-point.*

*Proof.* We will induct on the degree $d = [K : k]$. When $d = 1$ the result is vacuous. Suppose we know the claim for all fields of odd degree $d' < d$. We may assume that $K = k(\alpha)$ for some $\alpha$. This is always the case if $K/k$ is separable. In general, if $K = k(\alpha_1, \ldots, \alpha_r)$ then we apply the inductive hypothesis to the intermediate extensions $k(\alpha_1, \ldots, \alpha_{i+1})/k(\alpha_1, \ldots, \alpha_i)$ which have odd degree $< d$ to produce a rational point.

Now write the $K$-point of $X$ as $p = [g_0(\alpha)\colon \ldots \colon g_n(\alpha)]$ where $g_i$ are polynomials over $k$ of degree $\leq d-1$. Then $p$ lies on the rational curve $C$ of degree $m \leq d-1$ given by the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^n \qquad z \mapsto [g_0(z)\colon \ldots \colon g_n(z)].$$

If $C \subset X$, then $X$ contains a rational point by Lemma 1.10. Otherwise, the intersection $C \cap X$ is a zero dimensional degree $2m$ $k$-scheme which contains $p$ as a $K$-point. The Galois orbit of $p$ gives a degree $d$ $k$-subscheme of $C \cap X$ and the residual intersection if a degree $2m-d$ subscheme. Since $2m - d \leq d - 2 < d$ is odd, then $X$ has an odd degree point for some $d' < d$ so $X$ has a $k$-point by induction.

$\square$

## 3.2 Quadric bundles

The following relative situation will come up often.

**Definition 3.4.** A quadric bundle is a flat projective morphism $f : X \to S$ of integral $k$-varieties such that the generic fiber of $f$ is a smooth quadric hypersurface over $k(S)$.

*Remark* 3.5. A quadric bundle of relative dimension 1 is often called a conic bundle.

First, we need the following Lemma.

**Lemma 3.6.** *Suppose $f : X \to S$ is a dominant morphism of varieties with integral generic fiber $X_{k(S)}$. If $X_{k(S)}/k(S)$ is rational and $S/k$ is rational then $X/k$ is rational.*

*Proof.* By spreading out the birational parametrization $\mathbb{P}^n_{k(S)} \dashrightarrow X_{k(S)}$, we get a birational map $\mathbb{P}^n \times U \to f^{-1}(U)$ where $U \subset S$ is a dense open. Since $S$ is rational, $U$ is birational to $\mathbb{P}^m$ so $f^{-1}(U)$ is birational to $\mathbb{P}^n \times \mathbb{P}^n$ as required. $\square$

**Proposition 3.7.** *Let $f : X \to S$ be a quadric bundle over a rational base $S$. If $f$ has an odd-degree rational multisection, then $X$ is rational.*

*Remark* 3.8. A rational multisection is a $Y \subset X$ such that $Y \to S$ is generically finite.

*Proof.* Without loss of generality, we may suppose that the generic fiber of $Y \to S$ is irreducible. Then by pulling back to $k(S)$, we obtain an odd degree point $Y_{k(S)}$ of $X_{k(S)}$. By Springer's Theorem 3.3, the conic $X_{k(S)}$ has a rational point and we conclude that $X_{k(S)}$ is rational. Then by Lemma 3.6, $X/k$ is rational. $\square$

**Corollary 3.9.** *Let $f : X \to S$ be a quadric bundle. Suppose there exists a unirational variety $Y$ and map $g : Y \to X$ such that $f \circ g : Y \to S$ is dominant. Then $X$ is unirational.*

*Proof.* Consider the pullback diagram

$$\begin{array}{ccc} Z \subset Y \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \longrightarrow & S \end{array} \tag{6}$$

where $Z$ is the reduced component of $Y \times_S X$ dominating $X$ and $Y$. Now $Z \to Y$ is a quadric bundle with section and $Z \to X$ is dominant so it suffices to prove that $Z$ is unirational. Thus without loss of generality we may assume that $S$ is unirational and $f$ has a section.

In that case, we consider again the same Diagram 6 where $Y \to S$ is a unirational parametrization of $S$. Then $Z \to Y$ is a quadric bundle with section over a rational variety so $Z$ is rational by Proposition 3.7 so $Z \to X$ induces a unirational parametrization of $X$.

$\square$

*Remark* 3.10. A theorem of Lang [35, Corollary to Theorem 6] says that a quadric bundle $f : X \to S$ of relative dimension $r$ over an $n$-dimensional base $S$ has a section if $r > 2^n - 2$.

## 3.3 Cubic hypersurfaces

Next we move on to discuss some rationality constructions for cubic hypersurfaces $X \subset \mathbb{P}^n$, that is, those defined by a degree 3 equation.

**Proposition 3.11.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface over a field with $\mathrm{char}(k) = 0$ and suppose that $X$ contains a line $L$. Then $X$ is unirational.*

*Proof.* Consider the projection $\pi_L : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-1}$ away from $L$. We can resolve this rational map by blowing up to get a morphism $\rho\colon \mathrm{Bl}_L\mathbb{P}^{n+1} \to \mathbb{P}^{n-1}$ with exceptional divisor $E \to L$ a $\mathbb{P}^{n-1}$ bundle over $L$. Then $\mathbb{P}^{n-1}$ can be identified with the space of planes containing $L$ and $\rho$ is the universal family of planes.

Let $\tilde{X}$ denote the strict transform of $X$ and let $Z = \tilde{X} \cap E$ so that

$$\tilde{X} = \mathrm{Bl}_L X \to X$$

with exceptional divisor $Z \to L$. Since $Z \to L$ is a $\mathbb{P}^{n-2}$ bundle over the line $L$, $Z$ is rational.

We claim that the restriction $\pi := \rho|_{\tilde{X}} : \tilde{X} \to \mathbb{P}^{n-1}$ is a conic bundle. To prove this, it suffices to base-change to the algebraic closure $\bar{k}$. Now the fiber $\pi^{-1}([P])$ for $[P] \in \mathbb{P}^{n-1}$ is the residual to $L$ part of the intersection $X \cap P$. Since $X$ is a cubic then $P \cap X \subset P$ is a cubic plane curve which contains the line $L$ so for generic $P$, the residual intersection is a smooth conic. Thus $\pi$ is a conic bundle over $\mathbb{P}^{n-1}$. Moreover, the restriction $Z \to \mathbb{P}^{n-1}$ is dominant and $Z$ is rational so the conic bundle $\tilde{X}$ is unirational by Corollary 3.9. $\square$

Next we show that every cubic hypersurface over an algebraically closed field contains a line.

**Theorem 3.12.** *Suppose $k = \bar{k}$ and $\mathrm{char}(k) = 0$ and let $X \subset \mathbb{P}^{n+1}$ be a generic smooth cubic hypersurface of dimension $n$.*

1. *For $n = 2$, $X$ contains 27 lines.*

2. *For $n \geq 2$, the Fano variety of lines $F(X, 1)$ is smooth and $2n - 4$ dimensional.*

*Proof.* First we consider the correspondence $\mathbb{L} \subset \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3))) \times \mathbb{G}\mathrm{r}(1, n+1)$ consisting of pairs $(X, L)$ where $L \subset \mathbb{P}^{n+1}$ is a line, $X \subset \mathbb{P}^{n+1}$ is a cubic hypersurface, not necessarily smooth or irreducible, such that $L \subset X$. The projection

$$\pi_2 : \mathbb{L} \to \mathbb{G}\mathrm{r}(1, n+1)$$

has fiber over a line $L$ given by

$$\pi_2^{-1}(L) = \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3) \otimes \mathscr{I}_L))$$

where $\mathscr{I}_L$ is the ideal sheaf of $L$. Using the ideal sequence

$$0 \to \mathscr{I}_L \to \mathscr{O}_{\mathbb{P}^{n+1}} \to \mathscr{O}_L \to 0$$

and the vanishing $H^1(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3) \otimes \mathscr{I}_L) = 0$, we see that $\pi_2^{-1}(L)$ has dimension

$$\binom{n+3}{3} - 5$$

for *every* line $L$. We conclude that $\mathbb{L}$ is irreducible since $\pi_2$ is a proper morphism with non-empty irreducible fibers of the same dimension. Moreover, $\pi_2$ is flat since the Hilbert polynomial of the fibers is constant. Since $\pi_2$ is a flat morphism with smooth fibers, it is a smooth morphism and thus $\mathbb{L}$ is smooth.

Moreover, $\dim \mathbb{G}\mathrm{r}(1, n+1) = 2n$ so

$$\dim \mathbb{L} = \binom{n+3}{3} + 2n - 5$$

If we can show that the first projection $\pi_1 : \mathbb{L} \to \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3)))$ is surjective, then we can conclude that for generic $X = \{f = 0\}$, the the fiber

$$\pi_1^{-1}(f) = F(X, 1)$$

is irreducible of dimension

$$\dim \mathbb{L} - \dim \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3))) = \binom{n+3}{3} + 2n - 5 - \binom{n+3}{3} - 1 = 2n - 4.$$

First we consider the case $n = 2$ so $X$ is a cubic surface. In that case, $2n - 4 = 0$ so $\dim \mathbb{L} = \dim \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3)))$. Since $\mathbb{L}$ is irreducible, its dimension is $\dim F + \dim Z$ where $F$ is a general fiber of $\pi_1$ and $Z$ is the image of $\pi_1$. Now we can check by hand that Fermat cubic

$$X = \{x_0^3 + x_1^3 + x_2^3 + x_3^3\}$$

has finitely many lines, in fact 27, so by upper-semicontinuity of fiber dimension, $\dim F = 0$ so $\dim Z = \dim \mathbb{L} = \dim \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(3)))$ and $\pi_1$ is surjective and we conclude that every cubic surface contains a line, and the generic one contains at most 27 lines.

For $n > 2$, we can intersect $X$ with a generic 3-plane $P$. By Bertini's theorem $X \cap P \subset P$ is a smooth cubic surface which contains a line. Thus $X$ contains a line and $\pi_1$ is surjective for all $n$. By generic smoothness, we conclude that $F(X, 1)$ is smooth for a general $X$.

Note this only shows that a smooth cubic surface contains at most 27 lines. In fact one can show using deformation theory that $\pi_1$ is a smooth morphism over the locus where $X$ is a smooth cubic hypersurface. This boils down to the computation

$$H^1(X, \mathscr{N}_{L/X}) = 0$$

where $\mathscr{N}_{L/X}$ is the normal bundle of the line $L \subset X$. To compute this we consider the normal bundle sequences for $L \subset X$, $X \subset \mathbb{P}^{n+1}$ and $L \subset \mathbb{P}^{n+1}$. Putting them together we get an induced exact sequence

$$0 \to \mathscr{N}_{L/X} \to \mathscr{N}_{L/\mathbb{P}^{n+1}} \to \mathscr{N}_{X/\mathbb{P}^{n+1}}\big|_L \to 0.$$

We have isomorphisms $\mathscr{N}_{L/\mathbb{P}^{n+1}} \cong \mathscr{O}_L(1)^{\oplus n}$ and $\mathscr{N}_{X/\mathbb{P}^{n+1}}\big|_L = \mathscr{O}_L(3)$. Then $\mathscr{N}_{L/X}$ is a line bundle on $L$ of rank $n-1$ and degree $n-3$. Moreover, if we write

$$\mathscr{N}_{L/X} \cong \mathscr{O}_L(a_1) \oplus \ldots \oplus \mathscr{O}_L(a_{n-1})$$

with $a_1 \leq a_2 \leq \ldots \leq a_{n-1}$, we must have $a_i \leq 1$ for all $i$ since they each have a nonzero map $\mathscr{O}_L(1)$.

When $n = 2$, the only possibility is $\mathscr{N}_{L/X} \cong \mathscr{O}_L(-1)$. In this case, $H^0(\mathscr{N}_{L/X}) = H^1(\mathscr{N}_{L/X}) = 0$ so we conclude that $\pi_1$ is finite étale over the locus where $X$ is smooth and every smooth $X$ contains the same number of lines, namely 27. In general, we have two options:

$$\mathscr{N}_{L/X} \cong \mathscr{O}_L(-1) \oplus \mathscr{O}(1)^{\oplus n-2}$$
$$\mathscr{N}_{L/X} \cong \mathscr{O}_L \oplus \mathscr{O}_L \oplus \mathscr{O}(1)^{\oplus n-3}.$$

In each case, $H^1(\mathscr{N}_{L/X}) = 0$ so $\pi_1$ is smooth whenever $X$ is smooth and we conclude that $F(X,1)$ is smooth of the expected dimension.

$\square$

Kollár has proved the following vast generalization.

**Theorem 3.13** ([31])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface over a field $k$. Suppose either $k$ is perfect with $\mathrm{char}(k) = 2$, or $\mathrm{char}(k) \neq 2$. Then $X$ is unirational if and only if $X(k) \neq \emptyset$.*

*Proof.* If $X$ is unirational then $X(k) \neq \emptyset$ by Lemma 1.10.

Conversely, let $p \in X(k)$ be a rational point and let $T_p \subset \mathbb{P}^{n+1}$ be the tangent hyperplane $T_pX$ to $p$. The intersection $T_p \cap X = C_p$ is a cubic in $T_p$ with multiplicity 2 at $p$. In particular, $C_p$ is not a cone. Therefore, the projection

$$\pi_p : C_p \dashrightarrow \mathbb{P}^{n-1}$$

away from $p$ is birational so $C_p$ is a rational singular cubic. Let $\tau_p$ denote the rational inverse of $\pi_p$.

If $X$ contains two rational points $p$ and $q$, then we have a rational map

$$\Phi_{p,q} := f \circ (\tau_p \times \tau_q) : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \dashrightarrow X$$

where $f : C_p \times C_q \dashrightarrow X$ is given by the "third point" construction. That is, for general points $x \in C_p$ and $y \in C_q$, we consider the line $l_{x,y}$ through $x$ and $y$. Then the intersection

$$l_{x,y} \cap X = \{x, y, z\}$$

for some third point $z$ as long as $l_{x,y} \not\subset X$. The map $f$ is given by $f(x,y) = z$. Thus if $X$ has two general enough rational points, $\Phi_{p,q}$ is a unirational parametrization.

If $X$ does not have two rational points, we need to use Weil restriction of scalars to extend the construction.

**Definition 3.14.** Let $S' \to S$ be a morphism of schemes. For any $X'/S'$, the restriction of scalars $R_{S'/S}(X')$ is the functor $Sch_S \to Set$ given by

$$R_{S'/S}(X')(T) = \mathrm{Hom}_{S'}(T \times_S S', X')$$

We need to envoke the following fact about Weil restriction.

**Theorem 3.15.** *If $S' \to S$ is a finite flat morphism and $X'/S'$ is quasi-projective, then $R_{S'/S}(X')$ is representable by a scheme $\mathrm{R}_{S'/S}(X')$. If $S' \to S$ is a finite Galois extension $k'/k$, then $\mathrm{R}_{S'/S}(X')$ is the $S$-scheme obtained via descent from $\prod X'_\sigma$ where the product runs over all $\sigma \in \mathrm{Gal}(k'/k)$ and $X'_\sigma$ is the pullback of $X'$ along $\sigma : k' \to k'$.*

Note in particular that $\mathrm{R}_{k'/k}(X') \times_k k' \cong \prod X'_\sigma$ and by descent for morphisms, to produce a morphism $\mathrm{R}_{k'/k}(X') \to X$, it suffices to produce a Galois equivariant morphism

$$\prod_{\mathrm{Gal}(k'/k)} X'_\sigma \to X \times_k k'.$$

Now if $p$ is a $k'$ point of $X$ where $k'/k$ is separable of degree 2, let $\bar{p}$ denote the Galois conjugate of $p$. Then we have a $k'$ map

$$\Phi_{p,\bar{p}} : \mathbb{P}^{n-1}_{k'} \times \mathbb{P}^{n-1}_{k'} \dashrightarrow X_{k'}$$

via the third point construction and this map is Galois equivariant where the Galois actions swaps the copies of $\mathbb{P}^{n-1}$ and conjugates points. Thus it descends to a morphism of $k$-schemes

$$\varphi_p : \mathrm{R}_{k'/k}(\mathbb{P}^{n-1}_{k'}) \dashrightarrow X.$$

If $\Phi_{p,\bar{p}}$ is dominant then so is $\varphi_p$. Moreover $\mathrm{R}_{k'/k}(\mathbb{P}^{n-1}_{k'})$ is birational to $\mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k$ so if $p$ can be chosen to make $\Phi$ dominant, we have a unirational parametrization.

Now, given a rational point $p \in X(k)$ and a line $p \in L \subset \mathbb{P}^{n+1}$ defined over $k$ containing $p$, suppose that $L$ is transverse to $X$. Then $L \cap X = \{p, q, \bar{q}\}$ where either $q$ and $\bar{q}$ are $k$-points or a pair of conjugate $k'$ points. In the first case, we have a rational map

$$\Phi_{q,\bar{q}} : \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k \dashrightarrow X$$

and in the second case we have

$$\varphi_q : \mathrm{R}_{k'/k}(\mathbb{P}^{n-1}_{k'}) \dashrightarrow X.$$

In either case, we have a rational map depending on $L$ with source birational to $\mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k$ which we denote $\Phi_L$.

We want to pick $L$ general enough so that $\Phi_L$ is dominant. For infinite $k$, we can do this but for finite $k$ it could happen that $\Phi_L$ fails to be dominant for all $k$-lines $L$. Thus we consider the universal case over the variety of lines through $p \in \mathbb{P}^{n+1}$. The space of lines through $p$ is $\mathbb{P}^n$ and we have a double cover $Z \to \mathbb{P}^n$ given by the incidence correspodence

$$Z = \{(x, L) \mid p \neq x \in L\} \subset X \times \mathbb{P}^n.$$

Then over $Z$ we have a universal third point map

$$\Phi_Z : \mathbb{P}^{n-1}_Z \times_Z \mathbb{P}^{n-1}_Z \dashrightarrow X_Z$$

which is equivariant for the Galois group of $k(Z)/k(\mathbb{P}^n)$ and thus descends to a rational map out of the Weil restriction

$$\mathrm{R}_{Z/\mathbb{P}^n}(\mathbb{P}^{n-1}) \dashrightarrow X.$$

Since $\mathrm{R}_{Z/\mathbb{P}^n}(\mathbb{P}^{n-1})$ is birational to $\mathbb{P}^n_k \times \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k$, we obtain a rational map

$$\Phi : \mathbb{P}^n_k \times \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k \dashrightarrow X$$

given by $\Phi(L, u, v) = \Phi_L(u, v)$ on $\bar{k}$ points.

To check this map is dominant, we may pass to the algebraic closure $\bar{k}$ and check that

$$\Phi \times_k \bar{k} : \mathbb{P}^n_{\bar{k}} \times \mathbb{P}^{n-1}_{\bar{k}} \times \mathbb{P}^{n-1}_{\bar{k}} \dashrightarrow X \times_k \bar{k}$$

is dominant. Over $\bar{k}$, the map is given by $\Phi(L, u, v) = \Phi_{q,r}(u, v)$ where $L \cap X = \{p, q, r\}$. In particular, $q$ and $r$ are not general, they are colinear with $p$ so we need a criteria for the third-point map to be dominant. This follows from the following proposition.

**Proposition 3.16.** *[31, Proposition 14 & Lemma 15] Let $X$ be a smooth cubic over $k = \bar{k}$.*

*(a) $C_x$ is irreducible with a double point singularity at $x$ for $x$ general, and*

*(b) if $C_x$ and $C_y$ are as in (a) with $x \notin C_y$ and $y \notin C_x$, then the third point map $\Phi_{x,y}$ is dominant.*

*Proof.* (a) The only way the assertion can fail is if $C_x$ is either reducible, or a cone. If $C_x$ is reducible then it must contain an $n-1$-plane and if $C_x$ is a cone then the fibers of the projection from $x$ exhibit an an $n-2$-dimensional family of lines in $C_x$ through $p$. In either case $C_x$ contains an $n-2$ dimensional family of lines. Thus it suffices to show that there is not an $n-2$ dimensional family of lines through each point of $X$.

Now we induct on dimension. If $n = 2$, then $X$ contains finitely many lines. Indeed $\mathscr{N}_{L/X} = \mathscr{O}_L(-1)$ so every line on $X$ is rigid. Thus $X$ contains no lines through a general point so $C_x$ is irreducible with a double point singularity at $x$. Now we induct on dimension and suppose for every smooth cubic in $\mathbb{P}^n$, $C_x$ is irreducible with a double point singularity at $x$ for general $x$. By Bertini's Theorem, a general hyperplane section of $X$ is a smooth cubic in $\mathbb{P}^n$, but if $X$ has an $n-2$ dimensional family of lines through each point, then $X \cap H$ would have an $n-3$ dimensional family of lines through each point, contradicting the inductive hypothesis.

(b) First note that $\Phi_{x,y}$ is well defined. Indeed $C_y$ is not contained in the tangent plane $T_x$ and $C_x$ is not contained in $T_y$ for degree reasons. Thus, there exists a point $v \in C_y$ such that the line $l_{x,v}$ is not contained in $X$ and not tangent to $X$ at $x$. Similarly, there is a $u \in C_x$ such that $l_{u,y}$ is not contained in $X$ and not tangent to $y$ at $Y$. Since the join variety is irreducible and being transverse is an open condition, we conclude that for a generic choice of $u \in C_x$ and $v \in C_y$, the line $l_{u,v}$ is transverse to $X$ and so the third-point map is well defined at $(u, v)$.

By a dimension count, the expected dimension of the fibers of $\Phi_{x,y}$ is $n-2$ so the only way $\Phi_{x,y}$ can fail to be dominant is if all nonempty fibers have strictly larger dimension. Thus it suffices to exhibit a nonempty fiber with dimension $n-2$. We can describe the fiber $\Phi^{-1}_{x,y}(z)$ using the projection

$$\pi_z : \mathbb{P}^{n+1} \dashrightarrow T_y.$$

We have $\pi_z(u) = v$ if and only if $\{u, v, z\}$ lie on a line. Therefore $\Phi^{-1}_{x,y}(z)$ is birational to the locus of $(u, v) \in C_x \times C_y$ such that $\pi_z(u) = v$. For a general choice of $z$, $\pi_z|_X$ is generically finite so the dimension of this locus is equal to

$$\dim C_y \cap \pi_z(C_x).$$

Now let $v \in C_y$ be a general smooth point of $C_y$ and pick a $z$ such that the projection $\pi_z(x) = v$. Then for this $z$, $C_y$ and $\pi_z(C_x)$ have different multiplicities at $v = \pi_z(x)$. Since they are also irreducible, this means that $C_y$ and $\pi_z(C_x)$ do not share any irreducible components and so

$$\dim \Phi^{-1}_{x,y}(z) = \dim C_y \cap \pi_z(C_x) = n-2$$

as required.

□

This completes the proof as long as the third-point map $\Phi$ is well defined. The only issue is if every line $L$ through $p$ meets $X$ in another point with multiplicity 2. This happens if and only if the projection $\pi_p : X \dashrightarrow \mathbb{P}^n$ is not separable which can happen only in characteristic 2. In this case, we have the following. We omit the proof.

**Proposition 3.17.** *[31, Corollary 20] Let $k$ be a perfect field of characteristic 2 and $X \subset \mathbb{P}^{n+1}$ a smooth cubic with rational point $x \in X(k)$. Then there exists a point $p \in X(k)$ such that the projection $\pi_p$ is separable.*

□

On the other hand, certain cubic hypersurfaces are rational.

**Proposition 3.18.** *Suppose $X \subset \mathbb{P}^{2m+1}$ is a cubic hypersurface containing two disjoint $m$-planes $P$ and $P'$. Then $X$ is rational.*

*Proof.* Generalizing the argument of Proposition 3.11, we can project away from $P$ to obtain a quadric bundle

$$\pi : \tilde{X} = \mathrm{Bl}_P X \to \mathbb{P}^m.$$

The base $\mathbb{P}^m$ is the space of $m+1$-planes containing $P$ and the fiber above the point corresponding to an $m+1$-plane $\Lambda$ is the residual quadric of the intersection $\Lambda \cap X$. Since $P'$ is disjoint from $P$, then $\Lambda \cap P'$ is a single point for generic $\Lambda$. Thus, $P'$ is a section of $\pi$ and we conclude that $X$ is rational by Proposition 3.7.

□

In particular, this gives a proof that smooth cubic surfaces over an algebraically closed field of characteristic 0 are rational since they contain two disjoint lines. Of course, its easy to check that a smooth cubic surface is rational using Castelnuovo's Theorem. Indeed $h^1(X, \mathscr{O}_X) = 0$ for any hypersurface in $\mathbb{P}^{n+1}$ for $n \geq 2$. By the adjunction formula,

$$\omega_X = \omega_{\mathbb{P}^{n+1}}(X)|_X = \mathscr{O}_X(-n+1).$$

When $n = 2$, we have $\mathscr{O}_X(-1) = \omega_X$ and $\mathscr{O}_X(-2) = \omega_X^{\otimes 2}$ which has no sections so $P_2(X) = 0$.

## 3.4 Hypersurfaces of low degree compared to the dimension

The ideas of the previous section can be pushed to prove the following theorem of Harris–Mazur–Pandharipande [25], building off of work of Paranjape–Srinivas [42] and others which dealt with the case of a general hypersurface or complete intersection.

**Theorem 3.19.** *Let $k = \bar{k}$ and $\mathrm{char}(k) = 0$ and suppose $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d$. If $n \gg d$, then $X$ is unirational.*

As in Proposition 3.11, the idea of the proof is to produce an linear subspace contained in $X$ and consider the projection $\pi : X \dashrightarrow \mathbb{P}^m$. Then $\pi$ is a fibration by degree $d-1$ hypersurfaces in a smaller projective space. If $n \gg d$, then we can hope to induct on dimension and apply the result to the generic fiber of $\pi$. One major difficulty is that the generic fiber is defined over the non-algebraically closed function field $K$ of the base.

In order to actually produce the appropriate linear subspace contained in $X$, one needs to generalize the argument of Theorem 3.12 to show that the Fano variety $F(X, k)$ of $k$-planes of $X$ is of the expected dimension in a certain range for $k, n$ and $d$. This quite a subtle problem on its own. The state of the art in [5] yields an explicit bound of $n \geq 2^{d!} - 1$ in Theorem 3.19.

# 4 Rational curves on varieties

In this section we will study deformation spaces of rational curves on varieties. Our motivation is to develop the tools we need to study rationally connected (Section 5) and show that Fano varieties are rationally connected (Section 6). As a warm-up, we will first use the tools of deformation theory to study uniruled varieties.

**Definition 4.1.** A smooth projective variety $X/k$ is *uniruled* if there exists a rational curve through a general point. That is, for a general point $x \in X$, there exists a map $f : \mathbb{P}^1 \to X$ such that $f(\infty) = x$.

## 4.1 The Hom scheme

In this section we collect some results on the Hom scheme.

**Definition 4.2.** Let $X, Y \to S$ be schemes over $S$. The Hom functor $\mathscr{H}om_S(X, Y) : Sch_S \to Set$ is defined by

$$\mathscr{H}om_S(X, Y)(T) = Hom_T(X_T, Y_T).$$

For $B \subset X$ a closed subscheme and $g : B \to Y$ an $S$-morphism, we can define the closed subfunctor $\mathscr{H}om_S(X, Y, g)$ defined as the fiber product

$$\begin{tikzcd} \mathscr{H}om_S(X,Y,g) \arrow[r] \arrow[d] & S \arrow[d, "{[g]}"] \\ \mathscr{H}om_S(X,Y) \arrow[r] & \mathscr{H}om_S(B,Y) \end{tikzcd}$$

where the vertical map classifies the $S$-map $g : B \to Y$ and the bottom map is induced by composition $B \to X \to Y$. Concretely, we have

$$\mathscr{H}om_S(X, Y, g)(T) = \{f : X_T \to Y_T \mid f|_{B_T} = g_T\}.$$

Using the theory of Hilbert schemes (see e.g. [30, Chapter I]), we can prove the following.

**Theorem 4.3.** *Let $S$ be locally Noetherian, $X, Y \to S$ flat and quasi-projective $S$-schemes, and $B \subset X$ a closed subvariety flat over $S$. Then there exists a quasi-projective $S$-scheme*

$$\mathrm{Hom}_S(X, Y)$$

*representing the functor $\mathscr{H}om_S(X, Y)$. Moreover, for any $S$-map $g : B \to Y$, there exists a closed subscheme $\mathrm{Hom}_S(X, Y, g) \subset \mathrm{Hom}_S(X, Y)$ representing $\mathscr{H}om_S(X, Y, g)$.*

Over $\mathrm{Hom}_S(X, Y)$ we have the universal morphism

$$X \times_S \mathrm{Hom}_S(X, Y) \to Y \times_S \mathrm{Hom}_S(X, Y)$$

which we can compose with the projection to $Y$ to obtain the evaluation map

$$\mathrm{ev} : X \times_S \mathrm{Hom}_S(X, Y) \to Y.$$

On $k$-points, we can write $\mathrm{ev}(x, f) = f(x)$.

Moreover, the formation of the Hom-scheme is compatible with base change. In particular, given any point $s : \operatorname{Spec} k \to S$, the fiber $\mathrm{Hom}_S(X,Y)_s$ of the Hom-scheme over $s$ is the Hom-scheme of the fibers:

$$\mathrm{Hom}_S(X,Y)_s \cong \mathrm{Hom}_k(X_s, Y_s).$$

Fix a $k$-point $s : \operatorname{Spec} k \to S$ and suppose that $X_s$ is reduced and $Y_s$ is smooth. Then the infinitessimal structure of $\mathrm{Hom}_S(X,Y)$ at a $k$-point $f : X_s \to Y_s$ is controlled by a deformation-obstruction theory given by

$$H^i(X_s, f^*T_{Y_s}) \quad i = 0,1.$$

More generally, the deformation-obstruction theory for $\mathrm{Hom}_S(X,Y,g)$ at such an $f$ with $f|_{B_s} = g_s$ is given by

$$H^i(X_s, f^*T_{Y_s} \otimes \mathscr{I}_{B_s}) \quad i = 0,1$$

where $\mathscr{I}_{B_s}$ is the ideal sheaf of $B_s \subset X_s$.

**Theorem 4.4.** *[30, Theorems I.2.17 and II.1.7] Let $X, Y \to \operatorname{Spec} S$ and $g : B \to X$ as in Theorem 4.3. Fix $s : \operatorname{Spec} k \to S$ such that $S$ is equidimensional at $s$ and $X_s$ has no embedded points. Let $f : X_s \to Y_s$ be a morphism such that $Y_s$ is smooth in a neighborhood of the image of $f$.*

*(a)* *(i) The tangent space to $\mathrm{Hom}_k(X_s, Y_s)$ at $[f]$ is given by*

$$T_{[f]}\mathrm{Hom}_k(X_s, Y_s) = H^1(X_s, f^*T_{X_s}).$$

*(ii) Every irreducible component of $\mathrm{Hom}_S(X,Y)$ at $[f] : \operatorname{Spec} k \to \mathrm{Hom}_S(X,Y)$ has dimension*

$$\dim_{[f]} \mathrm{Hom}_S(X,Y) \geq h^0(X_s, f^*T_{Y_s}) - h^1(X_s, f^*T_{Y_s}) + \dim_s S.$$

*(iii) If*

$$\dim_{[f]} \mathrm{Hom}_k(X_s, Y_s) = h^0(X_s, f^*T_{Y_s}) - h^1(X_s, f^*T_{Y_s}),$$

*then $\mathrm{Hom}_S(X,Y) \to S$ is an lci morphism at $[f]$. In particular, $\mathrm{Hom}_S(X,Y)$ is flat over $S$ at $[f]$.*

*(iv) If $H^1(X_s, f^*T_{X_s}) = 0$, then $\mathrm{Hom}_S(X,Y)$ is smooth over $S$ of relative dimension $h^0(X_s, f^*T_{X_s})$ at $[f]$.*

*(b)* *(i) The tangent space to $\mathrm{Hom}_k(X_s, Y_s, g_s)$ at $[f]$ is given by*

$$T_{[f]}\mathrm{Hom}_k(X_s, Y_s) = H^1(X_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s}).$$

*(ii) Every irreducible component of $\mathrm{Hom}_S(X,Y,g)$ at $[f] : \operatorname{Spec} k \to \mathrm{Hom}_S(X,Y,g)$ has dimension*

$$\dim_{[f]} \mathrm{Hom}_S(X,Y,g) \geq h^0(X_s, f^*T_{Y_s} \otimes \mathscr{I}_{B_S}) - h^1(X_s, f^*T_{Y_s} \otimes \mathscr{I}_{B_s}) + \dim_s S.$$

*(iii) If*

$$\dim_{[f]} \mathrm{Hom}_k(X_s, Y_s, g_s) = h^0(X_s, f^*T_{Y_s} \otimes \mathscr{I}_{B_s}) - h^1(X_s, f^*T_{Y_s} \otimes \mathscr{I}_{B_s}),$$

*then $\mathrm{Hom}_S(X,Y,g) \to S$ is an lci morphism at $[f]$. In particular, $\mathrm{Hom}_S(X,Y,g)$ is flat over $S$ at $[f]$.*

*(iv) If $H^1(X_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s}) = 0$, then $\mathrm{Hom}_S(X,Y,g)$ is smooth over $S$ of relative dimension $h^0(X_s, f^*T_{X_s} \otimes \mathscr{I}_{B_s})$ at $[f]$.*

Our main case of interest will be when $X = C$ is a smooth (or more generally reduced Gorenstein) curve and $B \subset C$ is a collection of distinct smooth points. In this case we can use Riemann–Roch to compute the dimension bounds.

**Proposition 4.5.** *Let $C/k$ be a smooth projective curve, $B \subset C$ a collection of smooth points, and $f : C \to Y$ a morphism to some quasi-projective variety $Y/k$. Then*

$$\dim_{[f]} \operatorname{Hom}_k(C, Y) \geq -K_Y.C + d(1-g)$$
$$\dim_{[f]} \operatorname{Hom}_k(C, Y, f|_B) \geq -K_Y.C + d(1-g-n)$$

*where $g = g(C)$ is the genus, $n = \#B$ is the cardinality of $B$ and $d = \dim Y$.*

*Proof.* By Theorem 4.4, the lower bound on dimension is given by $\chi(f^*T_Y)$ (resp. $\chi(f^*T_Y(-B))$). Now $f^*T_Y$ is a rank $d$ vector bundle and $\det T_Y = \mathscr{O}_Y(-K_Y)$ so $\deg f^*T_Y = -K_Y.C$ (resp. $\deg f^*T_Y(-B) = -K_Y.C - n$. The result then follows by Riemann–Roch for vector bundles. $\square$

*Remark* 4.6. The above proposition holds as is for $C$ an integral Gorenstein curve if we take $g$ to be the arithmetic genus of $C$ and $B$ to be a collection of smooth points. We can try to drop irreducibility but then we have to be a bit careful about Riemann–Roch and computations of degrees.

## 4.2 Free rational curves

When $C = \mathbb{P}^1$ and $f : \mathbb{P}^1 \to X$ is a morphism, the local structure of $\operatorname{Hom}_k(\mathbb{P}^1, X)$ at $[f]$ is controlled by the splitting type $a_1 \geq a_2 \geq \ldots \geq a_d$ of $f^*T_X$:

$$f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(a_d).$$

In particular, $\operatorname{Hom}_k(\mathbb{P}^1, X)$ is smooth at $[f]$ if and only if $a_d \geq -1$. More generally, if $B \subset \mathbb{P}^1$ is a collection of $n$ smooth points and $g : B \to X$, then we need $a_d \geq n-1$ to guarantee smoothness of $\operatorname{Hom}_k(\mathbb{P}^1, X, g)$ at a point $f$. Note that $a_d \geq r$ if and only if $f^*T_X(-r)$ is generated by global sections.

**Definition 4.7.** A rational curve $f : \mathbb{P}^1 \to X$ is $r$-free for some integer $r \geq 0$ if $f^*T_X(-r)$ is generated by global sections. Similarly, $f$ is $r$-free relative to $B$ if $f^*T_X(-B-r)$ is generated by global sections.

When $r = 0$, $f$ is said to be a free rational curve and when $r = 1$, $f$ is said to be very free.

**Lemma 4.8.** *Suppose $X$ has a very free rational curve. Then $X$ has an $r$-free rational curve for all $r \geq 1$.*

*Proof.* By assumption, we have $f : \mathbb{P}^1 \to X$ such that $f^*T_X(-1)$ is globally generated. If we compose $f$ with a ramified $r$-fold cover $h : \mathbb{P}^1 \to \mathbb{P}^1$, we conclude that $(f \circ h)^*T_X(-r) = h^*(f^*T_X(-1))$ is globally generated so $f \circ h$ is $r$-free. $\square$

Our main interest in free rational curves is that they can easily be deformed to pass through points of $X$.

**Proposition 4.9.** *Fix an integer $s \geq 0$, a length $n$ subscheme $B \subset \mathbb{P}^1$, and $f : \mathbb{P}^1 \to X$ an $r$-free rational curve. If $n + s \leq r + 1$ then the evaluation map*

$$\begin{aligned} \mathrm{ev}_s : (\mathbb{P}^1)^s \times \operatorname{Hom}_k(\mathbb{P}^1, X, f|_B) &\to X^s \\ (t_1, \ldots, t_s, h) &\mapsto (h(t_1), \ldots, h(t_s)) \end{aligned}$$

*is smooth at $(t_1, \ldots, t_s, f)$ for $t_i \not\subset B$.*

*Proof.* Since $f$ is $r$-free and $n \leq r$, $H^1(\mathbb{P}^1, f^*T_X(-B)) = 0$ so $(\mathbb{P}^1)^s \times \mathrm{Hom}_k(\mathbb{P}^1, X, f|_B)$ is smooth at $(t_1, \dots, t_s, f)$. Thus it suffices to show that the differential of ev is surjective. The differential is given by

$$\mathrm{dev}_s : \bigoplus_{i=1}^{s} T_{\mathbb{P}^1, t_i} \bigoplus H^0(\mathbb{P}^1, f^*T_X(-B)) \to \bigoplus_{i=1}^{s} T_{X, f(t_i)} \tag{7}$$

$$(v_1, \dots, v_s, \partial) \mapsto (df_{t_1}(u_1) + \partial(t_1), \dots, df_{t_s}(u_s) + \partial(t_s)). \tag{8}$$

To show surjectivity, it suffices to show that for each $j$ and $u \in T_{X,f(t_j)} = (f^*T_X)_{t_j}$, there exists a section of $f^*T_X$ which vanishes at $B$ and at $t_i$ for $i \neq j$ and takes on value $u$ at $t_j$. Now $f^*T_X$ is $r$-free and $n + s \leq r + 1$ so

$$f^*T_X(-B - \sum_{i \neq j} t_i)$$

is globally generated which exactly means it admits a section taking on any value $u$ at $t_j$. Twisting back up gives us the required section taking value $u$ at $t_j$ and vanishing along $B$ and $t_i$ for $i \neq j$. $\square$

*Remark* 4.10. The above argument can be run in reverse using a Nakayama's lemma and a bit of the classification of bundles on $\mathbb{P}^1$ to see that if $[f]$ is a point of $\mathrm{Hom}(X, Y, f|_B)$ where $\mathrm{ev}_s$ is surjective at $(t_1, \dots, t_s, f)$ with $t_i \notin B$, then $f$ is $\min\{2, n + s - 1\}$-free (see [26, Proposition 4.3]).

Finally, we show that the

**Proposition 4.11.** *Let $X \to S$ be a smooth projective family of varieties over a locally Notherian $k$-scheme $S$. Suppose that $X_0$ admits an $r$-free rational curve for some $0 \in S$ and $r \geq 0$. Then there exists an open set $0 \in U \subset S$ such that $X_u$ admits an $r$-free rational curve for all $u \in U$.*

*Proof.* On $\mathbb{P}^1$, a vector bundle $\mathscr{E}$ is globally generated if and only if $H^1(\mathbb{P}^1, \mathscr{E}(-1)) = 0$. In particular, if $f : \mathbb{P}^1 \to X_0$ is an $r$-free rational curve, then $H^1(f^*T_{X_0}(-r-1)) = 0$.

Now we consider

$$\pi : H := \mathrm{Hom}_S(\mathbb{P}^1_S, X) \to S.$$

By Theorem 4.4, $\pi$ is smooth at $[f]$ over $0 \in S$. Over $H$, we have the universal morphism

$$\rho : \mathbb{P}^1_H \to X_H.$$

The bundle $\rho^*T_{X_H}(-r-1)$ is flat over $H$ and restricts to $h^*T_X(-r-1)$ for any point $[h] \in H$. In particular,

$$H^1(\rho^*T_{X_H}(-r-1)|_{[f]}) = H^1(f^*T_X(-r-1)) = 0.$$

By the Semi-Continuity Theorem, there exists an open neighborhood $[f] \in V \subset H$ such that $H^1(h^*T_X(-r-1)) = 0$ for all $[h] \in V$, that is, $h : \mathbb{P}^1 \to X_{\pi([h])}$ is an $r$-free rational curve for each $[h] \in V$. Up to shrinking $U$ and using the fact that $\pi$ is smooth at $[f]$, we may assume that $\pi|_V : V \to S$ is smooth. In particular, $\pi(V) = U \subset S$ is open and for all $u \in U$, there exists an $[h] \in V$ such that $h : \mathbb{P}^1 \to X_u$ is an $r$-free rational curve.

## 4.3 Uniruled varieties

We are now ready to prove the main result on uniruled varieties.

**Proposition 4.12.** *Let $X/k$ be a smooth projective variety. Then the following are equivalent.*

(1) *$X$ is uniruled.*

(2) *There exists $Y$ and a dominant map $\mathbb{P}^1 \times Y \dashrightarrow X$ that is not constant on $\mathbb{P}^1$.*

(3) *There exists a free rational curve $f : \mathbb{P}^1 \to X$.*

(4) *For a general point $x \in X$, there exists a free rational curve $f : \mathbb{P}^1 \to X$ with $f(\infty) = x$.*

*Proof.* (4) $\implies$ (3), (2) $\implies$ (1) and (4) $\implies$ (1) are clear.

For (1) $\implies$ (2), we use that the Hom-scheme is a countable union of quasi-projective schemes. Indeed, if we fix the degree $d$, the scheme $H_d := \mathrm{Hom}_k(\mathbb{P}^1, X, d)$ of degree $d$ maps is finite type. Since $X$ is uniruled, the evaluation map

$$\mathrm{ev} : \bigsqcup_d \mathbb{P}^1 \times H_d \to X$$

is dominant. Since $k$ is uncountable, we must have that $\mathbb{P}^1 \times H_d$ is dominant for some $d$. Since $H_d$ is of finite type, it must contain some irreducible component $Y$ such that $\mathbb{P}^1 \times Y \to X$ is dominant.

For (3) $\implies$ (4), suppose $f : \mathbb{P}^1 \to X$ is a free rational curve. Then by Proposition 4.9, the evaluation map

$$\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Hom}_k(\mathbb{P}^1, X) \to X$$

is smooth at $(t, [f])$, but $X$ is irreducible so ev is dominant. By the proof of Proposition 4.11, there exists a dense open set $[f] \in V \subset \mathrm{Hom}_k(\mathbb{P}^1, X)$ such that $h$ is free for any $[h] \in V$. Now the restriction $\mathrm{ev} : \mathbb{P}^1 \times V \to X$ is still dominant so for a general point $x \in X$, there exists $(t, [h]) \in \mathbb{P}^1 \times V$ such that $h : \mathbb{P}^1 \to X$ is free and $h(t) = x$.

The only thing left to prove is (1) $\implies$ (3). Since $\mathrm{ev}_{red} : \mathbb{P}^1 \times (H_d)_{red} \to X$ is dominant for some $d$ and we are in characteristic 0, then $\mathrm{ev}_{red}$ is smooth at a general point $(t, [f])$. Then

$$\mathrm{ev} : T_{\mathbb{P}^1,t} \oplus H^0(f^*T_X) \to T_{X,f(t)}$$

is surjective for general $(t, [f])$. Thus $H^0(f^*T_X) \to T_{X,f(t)}/df_t$ is surjective. Now $f^*T_X$ contains $T_{\mathbb{P}^1}$ which is generated by global sections so we have a commutative diagram

$$\begin{tikzcd}
H^0(\mathbb{P}^1, T_{\mathbb{P}^1}) \arrow[r] \arrow[d] & T_{\mathbb{P}^1,t} \arrow[d, "df_t"] \\
H^0(f^*T_X) \arrow[r] & T_{X,f(t)}
\end{tikzcd}$$

where the top map is surjective and the bottom map is surjective modulo $df_t$. We can conclude that the bottom map is surjective and thus $f^*T_X$ is globally generated at $t$ by Nakayama's lemma. Since $t$ is generic this implies that $f^*T_X$ is generically globally generated, but on $\mathbb{P}^1$ this is equivalent to globally generated. This is a special case of the argument alluded to in Remark 4.10. $\square$

As a consequence, we obtain that the property of being uniruled is deformation invariant.

**Theorem 4.13.** *Let $f : X \to S$ be a smooth projective morphism over a connected base $S$. Suppose for some $0 \in S$, $X_0$ is uniruled. Then for all $s \in S$, $X_s$ is uniruled.*

*Proof.* By Propositions 4.12 and 4.11, the property of being uniruled is open in $S$. Thus, it suffices to show that being uniruled is a closed in 1-parameter families.

Let $(C, 0) \to S$ be a smooth pointed curve mapping to $S$ with $U = C \setminus U$. Suppose that $X_u$ is uniruled for each $u \in U$ and let $x_0 \in X_0$ be a general point. Up to replacing $(C, 0)$ by a ramified cover, we may assume that there exists a section $x : C \to X_C$ with $x(0) = x_0$ and $x(u) = x_u \in X_u$ general. Since $X_u$ is uniruled for $u \in U$, there exists a map $f : \mathbb{P}^1 \times U \to X_U$ with $f(\infty, u) = x_u$ for some $t \in \mathbb{P}^1$. We can extend $f$ to a rational map

$$\mathbb{P}^1 \times C \dashrightarrow X_C.$$

After blowing up $\mathbb{P}^1 \times C$, we can resolve indeterminacies to obtain a smooth surface $Y$ and a diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & X_C \\ C & \end{tikzcd}$$

and a section $\infty : C \to Y$ given by the closure of $\infty \times U$. Now $Y \to \mathbb{P}^1 \times C$ is an iterated blowup at smooth points so the fiber $Y_0$ is a connected tree of rational curves with a marked point $\infty(0) \in Y_0$. Thus we have a map $g_0 := g|_{Y_0} : Y_0 \to X_0$ with $g(\infty(0)) = x_0$. It suffices to show that $g_0$ is nonconstant. Then there exists some component of $E \subset Y_0$ which is necessarily a smooth rational curve such that $g|_E : E \to X_0$ contains $x_0$. To deduce that $g_0$ is non-constant, we use the following Rigidity Lemma. $\square$

**Lemma 4.14** (Rigidity Lemma). *[33, Lemma 1.6] Let $f : Y \to Z$ be a proper surjective equidimensional morphism. Suppose that $Y$ is irreducible and $f$ has connected fibers. If $g : Y \to X$ is a morphism with $g|_{Y_0}$ constant for some $0 \in Z$. Then $g$ is constant on every fiber of $f$.*

FInally, we have the vanishing of plurigenera for uniruled varieties.

**Proposition 4.15.** *Suppose $X$ is a smooth projective uniruled variety. The $P_m := h^0(X, \omega_X^{\otimes m}) = 0$ for all $m \geq 1$.*

*Proof.* Let $\omega \in h^0(X, \omega_X^{\otimes m})$ be a section. We wish to show $\omega$ is 0. For a general $x \in X$, there exists a free rational curve $f : \mathbb{P}^1 \to X$ through $x$. But then $f^*T_X$ contains only summands of non-negative degree but it also contains a summand of degree at least 2 given by the existence of the differential $df : T_{\mathbb{P}^1} \to f^*T_X$. Thus, $\deg f^*T_X = -K_X.f(\mathbb{P}^1) > 0$ and so $mK_X.f(\mathbb{P}^1) < 0$ for all $m \geq 1$. This implies that $\omega_X^{\otimes m}|_{\mathbb{P}^1}$ is not effective so $\omega|_{\mathbb{P}^1} = 0$. But $x \in f(\mathbb{P}^1)$ was a general point so $\omega(x) = 0$ for a general point $x \in X$ from which we can conclude that $\omega = 0$. $\square$

One of the biggest conjectures in birational geometry states that the converse also holds.

**Conjecture 4.16** (Mori). *Suppose $P_m = 0$ for all $m \geq 1$. Then $X$ is uniruled.*

We end this section with the statement of the theorem of Boucksom–Demailly–Păun–Peternell.

**Theorem 4.17.** *[7] Let $X/\mathbb{C}$ be smooth projective. Then $X$ is uniruled if and only if $K_X$ is not pseudoeffective.*

Recall that a divisor is said to be pseudoeffective if it lies in the closure of the cone of effective divisors in $NS(X)_{\mathbb{R}}$. Note that $P_m = 0$ for all $m \geq 1$ if and only if $K_X$ is not in the effective cone. It is expected that $K_X$ is pseudoeffective if and only if it is effective. This is the so-called

Non-Vanishing Conjecture from the MMP. Thus Mori's conjecture is implied by the Non-Vanishing Conjecture.

The idea of the proof of Theorem 4.17 is the following. A pseudoeffective divisor $D$ is by definition the limit $\lim_{n\to\infty} D_n$ of effective divisors $D_n$. Now if $D_n$ is effective and

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ S & \end{tikzcd}$$

is a family of curves parametrized by $S$ which cover $X$, then $D_n.C_s \geq 0$ for all $n$ and $s \in S$. Taking limits, we see that $D.C_s \geq 0$ for all $s \in S$. The curves $C_s$ appearing in such a covering family are called *movable*.

In [7], it is shown that the converse also holds, that is, there is a duality between the cone of pseudoeffective divisors and the closure of the cone of movable curves. Namely, if $D.C_s \geq 0$ for every covering family $C \to S$ curves on $X$, then $D$ is pseudoeffective. In the special case that $D = K_X$, we see that $K_X$ is not pseudoeffective if and only if there exists a covering family with $K_X.C_s < 0$. By the adjunction formula, $K_X|_{C_s} = K_{C_s}$ has negative degree so $C_s$ is a rational curve for all $s \in S$. Thus $X$ is covered by rational curves, and in particular, it is uniruled. $\square$

# 5 Rationally connected varieties

Rationally connected varieties were introduced by Campana [8] and Kollár–Miyaoka–Mori [32] as a class of varieties between uniruled and unirational with very good properties. In this section we will use the results of the previous section to study rationally connected varieties. For the rest of this section, we work over $k = \mathbb{C}$ and assume $X/k$ is smooth projective unless otherwise indicated. Let us recall the definition.

**Definition 5.1.** Let $X/k$ be a smooth projective variety. We say that $X$ is *rationally connected* if for any two general points $x, y \in X$, there exists a map $f\colon \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$.

## 5.1 Characterizations of rationally connected varieties

We begin with the analogue of Proposition 4.12.

**Proposition 5.2.** *Let $X/k$ be a smooth projective variety. The following are equivalent.*

*(1) $X$ is rationally connected.*

*(2) There exists an irreducible variety $Y$ and a map $\mathbb{P}^1 \times Y \dashrightarrow X$ that is non-constant on $\mathbb{P}^1$ such that*

$$\mathbb{P}^1 \times \mathbb{P}^1 \times Y \dashrightarrow X \times X$$

*is dominant.*

*(3) There exists a very free rational curve $f : \mathbb{P}^1 \to X$.*

*(4) For a general collection of $r + 1$ points, there exists an $r$-free rational curve passing through those points.*

*Proof.* The proof is almost identical to the proof of Proposition 4.12 with small modifications. For (1) $\implies$ (2), we consider the evaluation map

$$\mathrm{ev}_2 : \bigsqcup_d \mathbb{P}^1 \times \mathbb{P}^1 \times H_d \to X.$$

For (1) $\implies$ (3) we have that $\mathrm{ev}_2 : (\mathbb{P}^1)^2 \times H_d \to X^2$ is smooth at some point $(t_1, t_2, [f])$ so we get a surjection

$$H^0(f^*T_X) \to T_{X,f(t_1)}/df_{t_1} \oplus T_{X,f(t_2)}/df_{t_2}.$$

Now for each $i$ we consider sections vanishing at $t_i$ to get a diagram

$$\begin{tikzcd} H^0(\mathbb{P}^1, T_{\mathbb{P}^1}(-t_i)) \arrow[r] \arrow[d] & T_{\mathbb{P}^1,t_j} \arrow[d, "df_{t_j}"] \\ H^0(f^*T_X(-t_i)) \arrow[r] & T_{X,f(t_j)} \end{tikzcd}$$

and conclude as before that $f^*T_X(-t_i)$ is generically globally generated, and thus globally generated.

For (3) $\implies$ (4), by Lemma 4.8 there is an $r$-free curve $f : \mathbb{P}^1 \to X$. By Proposition 4.9, the map

$$\mathrm{ev}_{r+1} : (\mathbb{P}^1)^{r+1} \times \mathrm{Hom}(\mathbb{P}^1, X) \to X^{r+1}$$

is smooth at $(t_1, \ldots, t_{r+1}, [f])$ but $X^{r+1}$ is irreducible to $\mathrm{ev}_{r+1}$ is dominant.

Finally (4) $\implies$ (1) is clear. $\square$

**Proposition 5.3.** *Suppose $X$ is rationally connected. Then we have*

$$P_{m,n}(X) := h^0(X, (\Omega_X^n)^{\otimes m}) = 0$$

*for all $m, n \geq 1$.*

*Proof.* We proceed as in Proposition 4.15. Let $\omega \in H^0(X, (\Omega_X^n)^{\otimes m})$. If $f : \mathbb{P}^1 \to X$ is a very free rational curve, then $f^*\Omega_X$ is a direct sum of negative line bundles so the restriction $f^*(\Omega_X^n)^{\otimes m}$ has no sections for any $m, n \geq 1$. Thus $f^*\omega = 0$ but very free curves cover $X$ so $\omega = 0$. $\square$

**Corollary 5.4.** *A rationally connected variety is simply connected.*

*Proof.* We will prove that the profinite completion $\pi_1(X)$ is 0 by showing that any finite étale cover of $X$ is trivial. A topological argument shows that in fact $pi_1(X)$ is finite which would complete the proof, but we omit that here.

Now let $\pi : Y \to X$ be a finite étale cover of degree $n$ from a connected variety $Y$. Since $\mathbb{P}^1$ is simply connected, for any rational curve $f : \mathbb{P}^1 \to X$, the pullback $\mathbb{P}^1 \times_X Y$ is a disjoint union of rational curves mapping isomorphically to $\mathbb{P}^1$. Picking a component of $\mathbb{P}^1 \times_X Y$, we have a commutative diagram

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "g"] \arrow[d] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f"] & X \end{tikzcd}$$

where the left vertical map is an isomorphism. Suppose $f$ is very free. Then $f^*T_X(-1)$ is globally generated but by commutativity of the diagram, this is the same as $g^*\pi^*T_X(-1)$. Since $\pi$ is étale, $\pi^*T_X = T_Y$ so we conclude that $g^*T_Y(-1)$ is globally generated and $g$ is a very free rational curve. By Proposition 5.2(3), $Y$ is rationally connected.

next note that if $X$ is rationally connected, then $\chi(\mathscr{O}_X) = 1$ since $h^i(X, \mathscr{O}_X) = h^0(X, \Omega_X^i) = 0$ by Proposition 5.3. Thus $\chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X) = 1$. On the other hand, $\chi(\mathscr{O}_Y) = n\chi(\mathscr{O}_X)$ for a finite étale cover of degree $n$, so $n = 1$ and $Y \to X$ is an isomorphism.

$\square$

## 5.2 Rationally chain connected varieties

We need to consider the following generalization of rational connectivity.

**Definition 5.5.** A smooth projective variety is *rationally chain connected (RCC)* if for any two general points $x, y$, there exists a pointed curve $(C, 0, \infty)$ and a map $f : C \to X$ such that $f(0) = x$ and $f(\infty) = y$, $C$ is a nodal tree of rational curves, and $0, \infty$ are in the smooth locus of $C$.

**Theorem 5.6.** *A smooth projective variety $X$ is rationally connected if and only if it is rationally chain connected.*

*Proof.* The only direction requiring proof is $(RCC) \implies (RC)$ so suppose $X$ is rationally chain connected. In particular this means $X$ is uniruled.

**Lemma 5.7.** *Let $X$ be a smooth projective uniruled variety. Then for a very general point $x \in X$, any rational curve through $x$ is free. There exists a nonempty locus $X_{free} \subset X$ which is a countable intersection of open subvarieties such that any rational curve through $x \in X_{free}$ is free.*

*Proof.* Consider the map $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, X) \to X$. Enumerate the countably many components $Z_i$ of $\mathrm{Hom}(\mathbb{P}^1, X)$. If $\mathbb{P}^1 \times Z_i \to X$ is not dominant, we let $U_i$ be the complement of its scheme theoretic image in $X$. If $\mathbb{P}^1 \times Z_i \to X$ is dominant, we consider the normalization $\tilde{Z} \to Z$ and induced map $\pi_i : \mathbb{P}^1 \times \tilde{Z}_i \to X$. By generic smoothness, there exists an open set $U_i \subset X$ such that $\pi_i^{-1}(U_i) \to U_i$ is smooth. In particular, the map on tangent spaces is surjective. This implies that the map on tangent spaces induced by $\mathbb{P}^1 \times Z_i \to X$ is surjective for any point lying over $U_i$. By Remark 4.10, every $f : \mathbb{P}^1 \to X$ parametrized by a point of $Z_i$ lying over $U_i$ is free. Now we let

$$X_{free} = \bigcap_i U_i.$$

Note that the argument applies even when $X$ is not uniruled but the result is vacuous since $X_{free}$ will simply be the locus of points through which there does not exist a rational curve. $\square$

Let $x \in X_{free}$ and $y \in X$ arbitrary. Then there exists a chain of rational curves connecting $x$ and $y$. Let us denote this chain by a collection of maps $f_i : \mathbb{P}^1 \to X$ for $i = 1, \dots s$ with $f_i(0) = x$, $f_s(\infty) = y$ and $f_0(\infty) = f_{i+1}(s)$ for $i < s$.

**Claim 5.8.** *We may suppose that $f_i$ meets $X_{free}$ for each $i$. In particular, $f_i$ is free.*

We prove the claim inductively. For $i = 1$, this is true by assumption, so suppose that $f_1, \dots, f_{i-1}$ meet $X_{free}$. Then we want to show that there exists a curve $g_i$ meeting $X_{free}$ with $g_i(0) = f_i(0)$ and $g_i(\infty) = f_i(\infty)$. In that case, we can replace $f_i$ with $g_i$ and by induction we obtain a chain connecting $x$ and $y$ with all curves free as claimed.

Since $f_{i-1}$ is free, the evaluation morphism

$$\mathrm{ev}_\infty : \mathrm{Hom}(\mathbb{P}^1, X) \to X$$

given by $[g] \mapsto g(\infty)$ is smooth at $[f_{i-1}]$. Let

$$T \subset \mathrm{ev}_\infty^{-1}(f_i(\mathbb{P}^1))$$

be the irreducible component containing $f_{i-1}$. Note that $T$ parametrizes maps $g : \mathbb{P}^1 \to X$ with $g(\infty)$ contained in the image of $f_i$. Since $\mathrm{ev}_\infty$ is smooth at $[f_{i-1}]$, $T$ dominates $f_i(\mathbb{P}^1)$. Now we consider the universal morphism over $T$

$$\begin{tikzcd} \mathbb{P}^1 \times T \arrow[r, "g"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$$

By Lemma 5.7, $X_{free}$ is a nonempty countable intersection of open sets. Since $T$ contains $g_{i-1}$ which meets $X_{free}$, then $g^{-1}(X_{free})$ is nonempty. Since $\pi$ is flat and thus open, $\pi(g^{-1}(X_{free}))$ is a nonempty countable intersection of open sets in $T$. Thus, the very general member of $T$ is a rational curve meeting $X_{free}$ and passing through $f_i(\mathbb{P}^1)$.

Now we build a map $f : C \to X$ from a comb $C$ with handle $C_0 \cong \mathbb{P}^1$ and teeth $E_1, \dots, E_k \cong \mathbb{P}^1$ such that $f|_{C_0} = f_i$ and $[f|_{E_i}] \in T$ and free. Since $T$ dominates $f_i(\mathbb{P}^1)$, we can choose $f|_{E_i}$ to pass through a very general point of $f_i(\mathbb{P}^1)$ and we can choose $k$ arbitrarily large.

**Proposition 5.9.** *Let $f : C \to X$ be a comb with handle $D$ and teeth $E_1, \dots, E_k$ and let $p_1, \dots, p_r \in D \cap C^{sm}$. Suppose that $f|_{E_k}$ is free for each $k$ and that*

$$k > K_X.f_*D + (r-1)\dim X + \dim_{f|_D} \mathrm{Hom}(\mathbb{P}^1, X, f|_{\{p_1,\dots,p_r\}}). \tag{9}$$

*Then there exists a smoothing of the restriction of $f$ to a sub-comb of $C$ that contains at least one tooth and that fixes $f(p_i)$.*

*Proof.* Let $\pi : \mathscr{C} \to S$ be a versal family of smoothings of $C$ over a $k$-dimensional base scheme $S \ni 0$ with $\mathscr{C}_0 = C$. Up to replacing $S$ by a cover, we can choose sections $\sigma_i : S \to \mathscr{C}$ such that $\sigma_i(0) = p_i \in C$. Let $g : \bigsqcup_i \sigma_i(S) \to X$ be the map which is constant equal to $f(p_i)$ on the section $\sigma_i(S)$. Consider the relative Hom-scheme

$$\pi : \mathrm{Hom}_S(\mathscr{C}, X_S, g) \to S$$

with universal map

$$F : \mathscr{C} \times_S \mathrm{Hom}_S(\mathscr{C}, X_S, g) \to X.$$

We show that $\pi$ is not constant in a neighborhood of $[f]$. By Theorem 4.4 and Proposition 4.5,

$$\dim_{[f]} \mathrm{Hom}_S(\mathscr{C}, X_S, g) \geq -K_X.(f_*D + \sum f_*E_i) + (1-r)\dim X + k.$$

Using inequality 9, we see that

$$\dim_{[f]} \mathrm{Hom}_S(\mathscr{C}, X_S, g) > -\sum K_X.f_*E_i + \dim_{f|_D} \mathrm{Hom}(\mathbb{P}^1, X, f|_{\{p_1,\dots,p_r\}}). \tag{10}$$

Now, $f|_{E_i} : E_i \to X$ is a free rational curve so

$$H^1(f|_{E_i}^* T_X(-q_i)) = 0$$

where $q_i = E_i \cap D$ is the point where $E_i$ meets $D$. Thus, $\dim_{f|_{E_i}} \mathrm{Hom}(E_i, X, f|_{q_i}) = -K_X.f_*E_i$. On the other hand, every morphism $g : C \to X$ is obtained by choosing a morphism $g_0 : D \to X$ and a collection of morphisms $g_i : E_i \to X$ with $g(q_i) = g_0(q_i')$ where $q_i' \in D$ is identified with $q_i \in E_i$. That is, there is a forgetful map

$$\mathrm{Hom}(C, X, f|_{\{p_1,\dots,p_r\}}) \to \mathrm{Hom}(D, X, f|_{\{p_1,\dots,p_r\}})$$

with fiber over $[g_0] \in \mathrm{Hom}(D, X, f|_{\{p_1,\dots,p_r\}})$

$$\prod_{i=1}^{k} \mathrm{Hom}(E_i, X, q_i \mapsto g_0(q_i')).$$

In particular, these fibers have constant dimension

$$-\sum K_X.f_*E_i$$

so $\dim_{[f]} \mathrm{Hom}(C, X, f|_{\{p_1,\dots,p_r\}}) = -\sum K_X.f_*E_i + \dim_{f|_D} \mathrm{Hom}(D, X, f|_{\{p_1,\dots,p_r\}})$. Putting this together with inequality 10, we have that

$$\dim_{[f]} \mathrm{Hom}_S(\mathscr{C}, X_S, g) > \dim_{[f]} \mathrm{Hom}(C, X, f|_{\{p_1,\dots,p_r\}}) = \dim_{[f]} \pi^{-1}(0)$$

and thus $\pi$ is not constant in a neighborhood of $[f]$.

Therefore we can pick a nontrivial 1-parameter family $T \to S$ over a smooth curve $T$ passing through 0 which lies in the image of $\pi$. Up to shrinking and taking a cover of $T$, we can lift this to a 1-parameter family

$$T \to \mathrm{Hom}_S(\mathscr{C}, X_S, g)$$

passing through $[f]$ and with generic fiber a nontrivial partial smoothing of $C$. More precisely, we have a family of maps

$$\begin{tikzcd} C \subset \mathscr{C}_T \arrow[r, "h"] \arrow[d] & X \\ 0 \in T & \end{tikzcd}$$

and sections $\sigma_i : T \to \mathscr{C}_T$ with $\sigma_i(0) = p_i$, $h|_{\mathscr{C}_0} = f$, $h(\sigma_i(T)) = f(p_i)$, and $\mathscr{C}_T \to T$ a nontrivial partial smoothing of $C$. In particular, $\mathscr{C}_T$ smooths at least one of the nodes of $C$. Let $C' \subset C$ be a maximal connected subcurve of $C$ containing $D$ and smoothed to a $\mathbb{P}^1$ by $\mathscr{C}_T \to T$. Then there exists a unique irreducible component $\mathscr{C}'_T \subset \mathscr{C}_T$ with generic fiber $\mathbb{P}^1$ and special fiber $C'$. Then $h|_{\mathscr{C}'_T}$ yields the desired smoothing of $f|_{C'}$. □

The above proposition is completely general, but in our setting, we also have the non-empty open set $X_{free}$ and as before, we can argue that a very general smoothing of $f|_{C'}$ meets $X_{free}$. Thus, we have constructed a smoothing $g_i$ of $f|_{C'}$ which meets $X_{free}$ and fixes $f_i(0)$ and $f_i(\infty)$ completing the inductive proof of the claim.

To summarize, we have seen that for $x \in X_{free}$ and $y \in X$, there exists a chain of free rational curves $f_i : \mathbb{P}^1 \to X$ with $f_1(0) = x$, $f_i(\infty) = f_{i+1}(0)$ for $i < s$, and $f_s(\infty) = y$. Now we use the following proposition which is similar to Proposition 5.9.

**Proposition 5.10.** *Let $f : C \to X$ be a chain of rational curves with $C = \cup_i C_i$ and $f|_{C_i} = f_i : C_i \to X$ for $i = 1, \ldots, s$. Let $\{p_1, \ldots, p_r\} \subset C^{sm}$ be a set of smooth marked points. Suppose that $C_i$ contains $r_i$ marked points, $f_s : C_s \to X$ is $r_s - 1$ free, and $f_i : C_i \to X$ is $r_i$ free for $i < s$. Then $f$ can be smoothed to an $r - 1$ free rational curve fixing $f(p_1), \ldots, f(p_r)$.*

*Proof.* We suppose that the points $p_1, \ldots, p_r$ are ordered so that the first $r_1$ are contained in $C_1$, the next $r_2$ are contained in $C_2$, etc.

Let

$$\begin{tikzcd} C \subset \mathscr{C} \arrow[d] \\ 0 \in T \end{tikzcd}$$

be a 1-parameter smoothing of $C$ over $(T, 0)$ a pointed smooth curve. Up to shrinking and taking a cover of $(T, 0)$ we can assume there exist disjoint sections $\sigma_i : T \to \mathscr{C}$ with $\sigma_i(0) = p_i$. Then we consider the relative Hom scheme

$$\pi : \mathrm{Hom}_T(\mathscr{C}, X_T, \{\sigma_i(T) \mapsto f(p_i)\}_{i=1,\ldots,r}\}) \to T.$$

We wish to show that $\pi$ is smooth in a neighborhood of $[f]$.

It suffices to show that $H^1(C, f^*T_X(-\sum p_i)) = 0$. We proceed by induction on $s$. If $s = 1$, then $C \cong \mathbb{P}^1$ and $f$ is $r-1$ free so $H^1(C, f^*T_X(-r)) = 0$. Now let $C' = C_2 \cup \ldots \cup C_s$ so that $C = C_1 \cup C'$ glued by identifying $q' \in C'$ and $q \in C_1$. Then for any vector bundle $\mathscr{E}$ on $C$, we have an exact sequence

$$0 \to \mathscr{E}|_{C_1}(-q) \to \mathscr{E} \to \mathscr{E}|_{C'} \to 0.$$

In particular, taking $\mathscr{E} = f^*T_X(-\sum p_i)$ we get

$$0 \to f^*T_X|_{C_1}(-p_1 - \ldots - p_{r_1} - q) \to f^*T_X(-\sum p_i) \to f^*T_X|_{C'}(-p_{r_1+1} - \ldots - p_r) \to 0.$$

Then $H^1(C_1, f^*T_X(-q-\sum_{i=1}^{r_1} p_i) = 0$ since $f_1$ is $r_1$-free and $H^1(C', f^*T_X|_{C'}(-\sum_{i=r_1+1}^{r} p_i) = 0$ by inductive hypothesis since $f|_{C'}$ satisfies the conditions of the proposition.

By the long exact sequence of cohomology, $H^1(C, f^*T_X(-\sum p_i)) = 0$ so $[f]$ is a smooth point of $\pi$. Thus up to further shrinking and taking a base change of $T$, we obtain a section of $\pi$ passing through $[f]$ by the lifting criterion of smoothness. Such a section is exactly a family of maps $F : \mathscr{C} \to X$ with $F(\sigma_i(T)) = f(p_i)$ and $F_0 = f$, that is, a smoothing of $f$ which fixes $f(p_i)$. Moreover, for $t \neq 0$ small enough, $H^1(\mathscr{C}_t, F_t^*T_X(-r)) = 0$ by semicontinuity of cohomology so $F_t$ is an $r-1$ free rational curve. $\square$

Applying Proposition 5.10 to the chain of free curves $f : C \to X$ connecting $x \in X_{free}$ with $y \in X$, we see that we can smooth $f$ into a free rational curve while fixing $y$. In particular, $x \in X_{free}$ is in the closure of the image of the evaluation map

$$\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, X, f(\infty) = y) \to X$$

but $X_{free}$ is dense in $X$ so ev must be dominant. In particular, through a generic $x \in X$, there exists a rational curve $f : \mathbb{P}^1 \to X$ with $f(0) = x$ and $f(\infty) = y$. Moreover, by generic smoothness, we can assume that $d\mathrm{ev}_{(t,f)}$ is surjective for generic $t \in \mathbb{P}^1$, that is,

$$T_{\mathbb{P}^1,t} \oplus H^0(f^*T_X(-1)) \to T_{X,f(t)}$$

is surjective. As before, this implies that $f^*T_X(-1)$ is generically globally generated since it contains a copy of $f^*T_{\mathbb{P}^1}(-1)$ which surjects onto $T_{\mathbb{P}^1,t}$ and thus the second component of the differential is already surjective. So $f^*T_X(-1)$ is globally generated on $\mathbb{P}^1$ and we conclude that $f$ is very free.

We conclude that for any $y \in X$ and general $x \in X$, there exists a very free rational curve $f : \mathbb{P}^1 \to X$ with $f(0) = x$ and $f(\infty) = y$. This concludes the proof that $X$ is rationally connected. $\square$

The proof of the above theorem actually gives us something stronger.

**Corollary 5.11.** *If $X$ is rationally connected, then through any collection of $r+1$ points $x_0, \ldots, x_r \in X$, there exists an $r$-free rational curve $f : \mathbb{P}^1 \to X$ and points $p_i \in \mathbb{P}^1$ with $f(p_i) = x_i$.*

*Proof.* When $r = 1$, we wish to show that any two points $x_0, x_1$ can be connected by a very free curve. The proof above shows that for a general point $y \in X$, there exists a very free rational curve $f_i : C_i \to X$ with $f(0) = x_i$ and $f(\infty) = y$. Gluing these together into a nodal curve $C = C_0 \cup C_1$, we get a chain $f : C \to X$ with two marked points $p_0, p_1 \in C$ such that $f(p_i) = x_i$ and each component is 1-free. By Proposition 5.10, $f$ can be smoothed into a 1-free rational curve while fixing $f(p_0)$ and $f(p_1)$.

In general, we let $f : C \to X$ be a chain of $r$ very free rational curves with marked points $p_i \in C$ such that $f(p_i) = x_i$. We can construct such a chain by first passing a very free rational curve through $x_0$ and $x_1$, then inductively on $r$, passing a very free rational curve through $x_r$ and a general point of an $r-1$ chain containing $x_0, \ldots, x_{r-1}$. Then by Proposition 5.10, $f$ can be smoothed into an $r$-free rational curve while fixing $f(p_i)$. $\square$

**Corollary 5.12.** *Let $f : X \to S$ be a smooth projective family over a connected base and suppose that $X_0$ is rationally connected for some $0 \in S$. Then $X_s$ is rationally connected for all $s \in S$.*

*Proof.* First, $X_0$ contains a very free rational curve and so does a small deformation of $X_s$. Thus the locus of $s \in S$ where $X_s$ is rationally connected is open. We wish to show it is also closed. In order to do that, we may suppose that $X \to T$ is a 1 parameter family with $X_t$ rationally connected for all $t \neq 0$. Taking the limit $\lim_{t\to 0}$ of a family of rational curves connecting $x_t, y_t \in X_t$ gives us a chain of rational curves connecting $x_0, y_0 \in X_0$. Up to shrinking and base extending $T$, any point $x \in X$ is the limit of a family of points $x_t \in X_t$. Thus, any two points of $X_0$ are connected by a chain of rational curves so $X_0$ is rationally chain connected and we conclude it is rationally connected by Theorem 5.6. □

## 5.3 MRC fibrations

**Theorem 5.13.** *Let $X$ be a smooth projective variety. There exists a smooth projective variety $Z$, dense open subsets $X_0 \subset X$ and and $Z_0 \subset Z$, and a rational map $\mu : X \dashrightarrow Z$ such that the following hold.*

1. *$\mu|_{X_0} : X_0 \to Z_0$ is a proper morphism.*
2. *the fibers of $\mu$ over $Z_0$ are rationally connected.*
3. *For a very general point $z \in Z$, a rational curve in $X$ meets the fiber $X_z$ if and only if it is contained in $X_z$.*

*Moreover, $\mu$ is unique up to birational equivalence.*

**Definition 5.14.** The variety $Z$ is called the maximally rationally connected (MRC) quotient of $X$, denoted by $R(X)$, and the map $\mu : X \dashrightarrow R(X)$ is the MRC fibration.

We won't give the full proof here but we will sketch the idea. Suppose

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ B & \end{tikzcd}$$

is a family of maps to $X$ parametrized by $B$. For us $\pi$ will be the universal family over some component of the space of rational curves. For any point $x \in X$, we can consider $V_1(x)$ the closure of $f(\pi^{-1}(\pi(f^{-1}(x))))$. This is the closure of the locus of points $x' \in X$ which are connected to $x$ by a curve $f_b : \mathscr{C}_b \to X$ for some $b$. Then for $k \geq 1$ we let

$$V_k(x) = \text{closure}(f(\pi^{-1}(\pi(f^{-1}(V_{k-1}(x)))))).$$

This is the closure of the locus of $x' \in X$ which are connected to $x \in X$ by a chain of $k$ rational curves in the family $\pi$. By dimension considerations the sequence $V_k(x) \subset V_{k+1}(x) \subset \ldots$ must stabilize for some $k$ large enough. Let $V_\infty(x)$ be this limit as $k \to \infty$. Then we have a rational map

$$X \dashrightarrow \text{Hilb}(X)$$

given by $x \mapsto [V_\infty(x)]$. We let $Z(\pi)$ be some smooth birational model of the image of this map. The rational map

$$X \dashrightarrow Z(\pi)$$

is birationally the quotient of $X$ by the equivalence relation $x \sim x'$ if $x$ and $x'$ can be connected by a chain of rational curves parametrized by $\pi$.

Now we apply this construction to the family

$$\begin{tikzcd} \mathscr{C}_m \arrow[r, "f_m"] \arrow[d, "\pi_m"'] & X \\ B_m \end{tikzcd}$$

of rational curves of degree at most $m$ to obtain a quotient $\mu_m : X \dashrightarrow Z_m$ of $X$ by the equivalence relation generated by chains of rational curves of degree at most $m$. Again the sequence of rational maps $\mu_m$ mus stabilize for dimension reasons to some $\mu_\infty : X \dashrightarrow Z_\infty$ and this is the MRC quotient. In particular, for very general $x \in X$, the fiber $\mu^{-1}(\mu(x))$ of the MRC quotient is exactly the rationally connected component of $x$:

$$\mu^{-1}(\mu(x)) = \{x' \in X \mid x' \text{ is connected to } x \text{ by a chain of rational curves}\}.$$

**Corollary 5.15.** *$X$ is rationally chain connected if and only of $R(X)$ is a point.*

**Theorem 5.16** (Graber–Harris–Starr [24])**.** *Let $f : X \to B$ be a surjective morphism from a smooth projective variety $X$ to a smooth projective curve $C$. If the general fiber of $f$ is rationally connected, then $f$ has a section.*

**Corollary 5.17.** *If $f : X \to B$ is a surjective morphism with rationally connected general fiber over a rationally connected base $B$, then $X$ is rationally connected.*

*Proof.* Let $x_1, x_2$ be general points with $f(x_i) = b_i$ Since $B$ is rationally connected, any two points $b_1, b_2 \in B$ can be connected by a rational curve $h : C \to B$ with $f(p_i) = b_i$. By GHS, $C \times_B X \to C$ has a section at least if we pick $f$ general enough. Let $g : C \to X$ denote the composition of this section with $C \times_B X \to X$. Then $g$ is a rational curve connecting $g(p_1)$ and $g(p_2)$ but since $x_i$ are general, the fibers $f^{-1}(b_i)$ are rationally connected so $g(p_i)$ can be connected to $x_i$ by a rational curve. Thus $x_1$ and $x_2$ are connected by a chain of rational curves so $X$ is rationally connected by Theorem 5.6. □

**Corollary 5.18.** *For any $X$, the MRC quotient $R(X)$ is not uniruled.*

*Proof.* Suppose $R(X)$ is uniruled (and in particular $\dim R(X) > 0$). Then there exists a rational curve through any point $z \in R(X)$ but by Theorem 5.16, this rational curve can be lifted to a rational curve in $X$ not contained in a fiber. This contradicts property (3) in the definition of the MRC fibration. □

**Conjecture 5.19** (Mumford)**.** *$X$ is rationally connected if and only if $P_{m,n}(X) = h^0(X, (\Omega_X^n)^{\otimes m}) = 0$ for all $m, n \geq 1$.*

**Corollary 5.20.** *Mori's conjecture 4.16 implies Mumford's conjecture 5.19.*

*Proof.* Suppose $X$ is not rationally connected. We will show that $P_{m,n}(X) \neq 0$ for some $m, n \geq 1$. Let $\mu : X \dashrightarrow R(X)$ be the MRC fibration. After resolving the rational map we may assume that we have a morphism $\mu' : X' \to Z$ between smooth projective varieties with $X' \to X$ birational. Moreover, since $P_{m,n}$ is a birational invariant, it suffices to prove the result for $X'$ so we may assume that $\mu$ is a morphism. By Corollary 5.17, $Z$ is not uniruled and by Corollary 5.18 $\dim Z > 0$. By Mori's conjecture, $h^0(Z, \omega_Z^{\otimes m}) \neq 0$ for some $m \geq 1$. Pulling back using generic smoothness and the birational invariance of $P_{m,n}$, we have an injection

$$H^0(\omega_Z^{\otimes m}) \hookrightarrow H^0((\Omega_X^{\dim Z})^{\otimes m})$$

so $P_{m,n}(X) \neq 0$ for some $m, n \geq 1$. □

The following result of Campana–Demailly–Peternell [43, 9] gives a characterization of rationally connected varieties analagous to the characterization of uniruled varieties in Theorem 4.17.

**Theorem 5.21.** *[9, 43] $X$ is rationally connected if and only if for some (resp. any) ample line bundle $A$ on $X$, there exist a constant $C_A > 0$ such that*

$$H^0(X, (\Omega^1_X)^{\otimes m} \otimes A^{\otimes k}) = 0$$

*for all $m \geq C_A k$.*

# 6 Fano varieties

In this section we study Fano varieties. Our main goal will be to show that Fano varieties are rationally connected.

**Definition 6.1.** A smooth projective variety is *Fano* if the anti-canonical divisor $-K_X$ is ample.

**Example 6.2.** By Example 2.12, a degree $d$ hypersurface $X \subset \mathbb{P}^n$ is Fano if and only if $d \leq n$. More generally, a complete intersection of type $(d_1, \dots, d_k)$ is Fano if and only if $\sum d_k \leq n$.

**Theorem 6.3.** *For $k = \mathbb{C}$, Fano varieties $X/k$ are rationally connected. For an algebraically closed field $k$ of any characteristic, Fano varieties $X/k$ are rationally chain connected, and in particular uniruled.*

In particular, hypersurfaces of degree $d \leq n$ are rationally connected. On the other hand, by Theorem 3.19 and the work of Beheshti–Riedl [5], we only know unirationality for much smaller degrees $2^{d!} - 1 \leq n$. In fact, it is expected that most Fano varieties are not unirational.

**Conjecture 6.4.** *There exist Fano hypersurfaces which are not unirational. In particular, a smooth hypersurface of degree $n$ in $\mathbb{P}^n$ is not unirational for $n \geq 5$.*

However, no examples of non-unirational degree $d \leq n$ smooth hypersurfaces are known.

## 6.1 Bend-and-break

In this section we prove the bend-and-break results of Mori [38] and its generalizations which allow us to produce rational curves by degenerating curves of very high anti-canonical degree. The proofs here are based on [20, Chapter 3].

**Proposition 6.5** (Bend-and-break I)**.** *Let $f : C \to X$ be a map from a smooth curve and fix a point $c \in C$. Suppose*

$$-K_X . f_*[C] - g \dim X > 0.$$

*Then there exists a rational curve on $X$ through $f(c)$.*

*Proof.* If $g(C) = 0$ we are done so suppose $g(C) \geq 1$. By the dimension bounds from Proposition 4.5, we know that

$$\dim \mathrm{Hom}(C, X, f|_c) \geq 1.$$

Let $T$ be a non-constant smooth curve on $\mathrm{Hom}(C, X, f|_c)$ passing through $f$ and let $\bar{T}$ be the compactification of $T$. The universal map of the Hom scheme gives us a rational map

$$F : C \times \bar{T} \dashrightarrow X$$

with $F|_{C_t} = f$ for some $t_0 \in T$ and $F(c, t) = f(c)$ for all $t \in T$.

Suppose that $F$ is a morphism in a neighborhood of $c \times \bar{T}$. Then $F$ contracts $c \times \bar{T}$ to a point so by the Rigidity Lemma 4.14, $F$ factors through the projection $C \times \bar{T} \to C$. Thus every map parametrized by $T$ has the same image in $X$. Since $g(C) > 0$, $\mathrm{Aut}(C, c) = 0$ so there are only finitely many maps with the same image. This contradicts that $T \to \mathrm{Hom}(C, X, f|_c)$ is non-constant. Thus $F$ has at least one base point of the form $(c, t_1)$.

Let $\mu : S \to C \times \bar{T}$ be a blowup of $C \times \bar{T}$ that resolves $F$ to a morphism $G : S \to X$ and let $E$ be the exceptional divisor over $(c, t_1)$. Then $E$ is a union of rational curves which is not contracted by $G$ to a point. Moreover, $G(E)$ meets $f(c)$ since $E$ meets the strict transform of $c \times \bar{T}$. Thus a component of $E$ which is not contracted by $G$ gives us a rational curve through $f(c)$. ☐

**Proposition 6.6** (Bend-and-break II)**.** *Suppose $f : \mathbb{P}^1 \to X$ is a rational curve with*

$$-K_X.f_*\mathbb{P}^1 - \dim X \geq 2.$$

*Then the cycle $f_*\mathbb{P}^1$ is algebraically equivalent to a connected reducible cycle of rational curves passing through $f(0)$ and $f(\infty)$.*

*Proof.* By assumption, the dimension of the space of maps

$$\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X, f|_{\{0,\infty\}}) \geq 2.$$

Since $\operatorname{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{G}_m$ is 1-dimensional, we can find a 1-parameter family

$$\begin{tikzcd} \mathbb{P}^1 \times T \arrow[r, "F"] \arrow[d] & X \\ T & \end{tikzcd}$$

of maps with $F_{t_0} = F_t$ for some $t_0$ and such that $T \to \operatorname{Hom}(\mathbb{P}^1, X, f|_{\{0,\infty\}})$ is not contained a $\mathbb{G}_m$ orbit. This implies that the image of $F_t$ is varying as $t \in T$ varies and so $F \times \mathrm{id}_T : \mathbb{P}^1 \times T \to X \times T$ is finite.

Let $\bar{T}$ be the smooth compactification of $T$ and let $S$ be the normalization of the closure of the image of $F \times \mathrm{id}_T$ in $K(\mathbb{P}^1 \times T)$. Then $S$ is finite over $X \times \bar{T}$ and we have a commutative diagram

$$\begin{tikzcd} \mathbb{P}^1 \times T \arrow[r] \arrow[dr, "F"'] & S \arrow[d] \\ & X \times \bar{T} \end{tikzcd} .$$

and a diagram

$$\begin{tikzcd} \mathbb{P}^1 \times T \arrow[r] \arrow[d] & S \arrow[d, "\pi"'] \\ T \arrow[r] & \bar{T} \end{tikzcd} .$$

Then $S$ is a normal surface fibered over $\bar{T}$ with a family of maps $\bar{F} : S \to X$ parametrized by $\bar{T}$ extending our family of maps $F$. Moreover, if $E \subset S \to \bar{T}$ is a component of a fiber of $\pi$, then $\bar{F}$ maps $E$ finitely into $X$. This follows since $S \to X \times \bar{T}$ is finite and $E$ is contracted by $\pi$ so it must not be contracted by the first projection. Moreover, $\pi : S \to \bar{T}$ is flat with generic fiber $\mathbb{P}^1$ so the fibers of $\pi$ are Cohen-Macaulay curves of arithmetic genus 0. It follows that each component of a reduced fiber of $\pi^{-1}(t)_{red}$ is a smooth rational curve. Moreover $\bar{F}|_{\pi^{-1}(t_0)} = f$. Thus, it suffices to show that there exists a reducible fiber of $S$, say over $t_\infty$. Then $\bar{F}(\pi^{-1}(t_\infty))$ gives us a reducible rational curve through $f(0)$ and $f(\infty)$ algebraically equivalent to $f_*\mathbb{P}^1$.

Suppose every fiber of $\pi$ is irreducible. Then every fiber is a $\mathbb{P}^1$. It follows that $\pi : S \to \bar{T}$ is a ruled surface over $\bar{T}$. Moreover, $\pi$ has two sections, the closures of $0 \times T$ and $\infty \times T$. Let us denote these $S_0$ and $S_\infty$. By construction, $\bar{F}(S_0) = f(0)$ and $\bar{F}(S_\infty) = f(\infty)$. Now $F$ is finite and thus $\bar{F}$ is generically finite. Therefore, by the Hodge Index Theorem, $S_i^2 < 0$ for $i = 0, 1$. On the other hand, if $S$ is a ruled surface with sections $S_0, S_\infty$, then $S_0 - S_\infty \sim f$ where $f$ is a fiber class of $\pi$. Thus

$$0 = (S_0 - S_\infty) = S_0^2 - 2S_0S_\infty + S_\infty^2 < 0.$$

This is a contradiction, so $\pi$ must have a reducible fiber.

□

Next we have the following relative versions. To state it we need some notation. Let $\pi : X \to Y$ be a morphism of projective varieties and let $f : C \to X$ a curve on $X$. Let $B \subset C$ be a collection of $n$ marked points and let $g = \pi \circ f : C \to Y$ be the composition. Then $\pi$ induces a map

$$\rho : \mathrm{Hom}(C, X, f|_B) \to \mathrm{Hom}(C, Y, g|_B).$$

We let $\mathrm{Hom}_Y(C, X, g, f|_B)$ denote the fiber

$$\rho^{-1}([g : C \to Y]).$$

This is the deformation space of maps $f' : C \to X$ with $f'|_B = f|_B$ and $\pi \circ f' = \pi \circ f = g$.

**Proposition 6.7** (Relative bend-and-break I)**.** *Let $\pi : X \to Y$ and $f : C \to X$ be as above. Suppose $g(C) > 0$ and fix $c \in C$. If*

$$\dim_{[f]} \mathrm{Hom}_Y(C, X, g, f|_c) \geq 1,$$

*then there exists $f' : C \to X$ and a connected effective rational 1-cycle $Z$ passing through $f(c)$ such that*

$$f_*C \sim_{alg} f'_*C + Z$$

*with $\pi \circ f' = \pi \circ f$ and $\pi(Z) = \pi(f(c))$.*

*Proof.* The proof proceeds as in the proof of Proposition 6.5. We have a 1-dimensional family of maps $F : C \times T \to X$. After compactifying $T \subset \bar{T}$ and resolving basepoints (which necessarily must exist along $c \times \bar{T}$), we have a surface $\mu : S \to C \times \bar{T}$ a morphism $G : S \to X$ which sends the connected component of the exceptional locus of $c \times \bar{T}$ to a connected effective rational 1-cycle $Z$ passing through $f(c)$. Moreover, we let $f' : C \to X$ be the restriction of $S \to X$ to the strict transform of $C$ on $S$ meeting $Z$.

The only thing to check is that $Z$ is contracted by $\pi$. To check this, note that the composition $C \times T \to X \to Y$ factors through the projection $C \times T \to C$ since by assumption all our maps lie over the fixed map $g : C \to X$. Thus the composition extends to a *morphism* $C \times \bar{T} \to X$. By commutativity, the composition $S \to X \to Y$ must be equal to $S \to C \times \bar{T} \to Y$ and so every exceptional curve of $S \to C \times \bar{T}$ is contracted on $Y$. Since $Z$ is the image of such curves, $Z$ is also contracted. ☐

**Proposition 6.8** (Relative bend-and-break II)**.** *Let $\pi : X \to Y$ be a morphism of projective varieties and $f : (C, c) \to X$ a smooth pointed curve mapping to $X$. If the map*

$$\rho : \mathrm{Hom}(C, X, f|_c) \to \mathrm{Hom}(C, Y, (\pi \circ f)|_c).$$

*does not contract every component of $\mathrm{Hom}(C, X, f|_B)$ containing $[f]$, then there exists a connected effective rational 1-cycle $Z$ passing through $f(c)$ such that $\pi|_Z$ is not constant.*

*Proof.* If $C$ is rational we are done so assume without loss of generality that $g(C) > 0$. Let $T \to \mathrm{Hom}(C, X, f|_c)$ be a curve through $[f]$ such that $\rho(T)$ is not a point. This is a family of maps $C \times T \to X$ fixing $c \times T$ such that the composition $C \times T \to Y$ is a non-isotrivial family of maps. In particular, as in the proof of Proposition 6.5, if $\bar{T}$ is a compactification of $T$, then the rational map $C \times \bar{T} \dashrightarrow Y$ must have a basepoint along $c \times \bar{T}$. Let $S \to C \times \bar{T}$ be a blowup resolving the indeterminacies of $C \times \bar{T} \dashrightarrow Y$ and $S' \to S$ a possibly further blowup resolving the indeterminacies of $C \times \bar{T} \dashrightarrow X$. Then we have a commutative diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & Y \end{tikzcd} .$$

Let $Z$ be the image under $S' \to Y$ of a connected component of the exceptional locus of $S' \to C \times \bar{T}$ which dominates an exceptional locus of $S \to C \times \bar{T}$ meeting $c \times \bar{T}$. Then $Z$ is an effective connected rational 1-cycle meeting $f(c)$ which is not contracted by $\pi$ by commutativity of the diagram. $\square$

## 6.2 Transverse rational curves on Fano varieties

Now we will use bend-and-break to produce rational curves on Fano varieties which are transverse to the fibers of a morphism. The basic idea, due to Mori, is to reduce to characteristic $p$ where we have the Frobenius morphism $F : C \to C$. The key point is that composing with $F$ raises the anticanonical degree of a map $f : C \to X$ without changing the genus of $C$, thus allowing us to end up in a situation where the bound

$$-K_X.f_*[C] - g \dim X > 0$$

is satisfied. This produces rational curves of arbitrary degree which then can be broken by bend-and-break II (Proposition 6.6) into rational curves of bounded degree. The boundedness of degrees is then crucial to allow us to lift back to characteristic 0. Finally, we need to carry out this argument in the relative setting to produce rational curves of bounded degree which are transverse to the fibers of a given morphism.

**Theorem 6.9.** *Let $X/k$ be a Fano variety with $k = \bar{k}$, $Y_0$ a quasi-projective variety with $\dim Y_0 \geq 1$, and $X_0 \subset X$ a dense open subset. Let $\pi : X_0 \to Y_0$ be a proper surjection. Then for any point $y \in Y_0$, there exists a rational curve $f : \mathbb{P}^1 \to X$ which meets $\pi^{-1}(y)$ but is not contained in $\pi^{-1}(y)$. Moreover, $f : \mathbb{P}^1 \to X$ can be chosen so that $-K_X.f_*\mathbb{P}^1 \leq \dim X + 1$.*

Let $X_0 = X = Y_0$ and $\pi$ to be the identity, we immediately get that Fano varieties are uniruled in any characteristic. Our goal is to prove that in fact they are rationally chain connected by applying the theorem to the MRC fibration $X \dashrightarrow R(X)$. For simplicity, we will only prove the theorem when $X_0 = X$ so that $f$ is a morphism. The general case is similar but where we have to resolve the morphism by blowing up $X$ and carefully keeping track of what happens (see [20, Theorem 3.22]).

*Proof.* Suppose first that $\mathrm{char}(k) = p > 0$. Since $\pi$ is projective, we can find a pointed curve $f : (C, c) \to X$ with $f(c) \in \pi^{-1}(y)$ but $f(C) \not\subset \pi^{-1}(y)$. If $C$ is rational we are done so suppose that $g(C) > 0$.

Let $H$ be an ample divisor on $Y$ and pick $0 < \alpha \ll 1$ small enough so that $-K_X - \alpha\pi^*H$ is ample. Let $F_m : C_m \to C$ be the $m$-fold Frobenius map. Then $g(C_m) = g(C) = g$ but $F_m$ has degree $p^m$ so

$$\alpha H.(\pi \circ f \circ F_m)_*C_m - g \dim X = \alpha p^m H.(\pi \circ f)_*C - g \dim X \geq 0$$

for $m$ large. Replacing $f$ with $f \circ F_m$, we may suppose without loss of generality that

$$\alpha H.(\pi \circ f)_*C = \alpha\pi^*H.f_*C \geq g \dim X. \tag{11}$$

It follows that

$$-K_X.f_*C > \alpha\pi^*H.f_*C \geq g \dim X. \tag{12}$$

At this point, already by bend-and-break I (Proposition 6.5), we have a rational curve through $f(c) \in \pi^{-1}(y)$ (and in particular we see that $X$ is uniruled). The issue is this rational curve might be contained in a fiber. However, relative bend-and-break II (Proposition 6.8) tells us that we may produce a rational curve not contained in the fiber as long as the composition map

$$\rho : \operatorname{Hom}(C, X, f|_c) \to \operatorname{Hom}(C, Y, (\pi \circ f)|_c)$$

does not contract every component of the source containing $[f]$.

Suppose for contradiction that $\rho$ does contract every component of $\operatorname{Hom}(C, X, f|_c)$. Then the image of $\rho$ on these components must be the point $[\pi \circ f]$. Therefore, in a neighborhood of $[f]$, $\operatorname{Hom}(C, X, f|_c) = \operatorname{Hom}_Y(C, X, (\pi \circ f), f|_c)$. Moreover, by equation (12),

$$\dim_{[f]} \operatorname{Hom}_Y(C, X, (\pi \circ f), f|_c) = \dim_{[f]} \operatorname{Hom}(C, X, f|_c) \geq 1$$

By relative bend-and-break II (Proposition 6.7), there exists $f' : C \to X$ and a connected effective rational 1-cycle $Z$ through $f(c)$ with

$$f_*C \sim_{alg} f'_*C + Z$$

such that $\pi(Z) = y$ and $\pi \circ f = \pi \circ f'$. In particular, $\alpha H.(\pi \circ f')_*C = \alpha H.(\pi \circ f)_*C$ so equation (11) still holds for $f' : C \to X$. On the other, the anticanonical degree satisfies

$$0 < -K_X.f'_*C < -K_X.f_*C$$

since $-K_X.Z > 0$. Thus, $f' : C \to X$ is a curve with strictly smaller anticanonical degree that satisfies (11). Thus we may repeat the argument to produce $f'' : C \to X$ with $0 < -K_X.f''_*C < -K_X.f'_*C$ still satisfying (11). In this way, we get an infinite decreasing sequence of positive integers which is impossible. Therefore, $\rho$ does not contract every component containing $[f]$ to a point so by relative bend-and-break II (Proposition 6.8), there exists a rational curve $Z$ passing through $f(c)$ such that $\pi|_Z$ is not constant.

Now let $f : (C, c) \to X$ be such a transverse rational curve with $f(c) \in \pi^{-1}(y)$ by $f(C) \not\subset \pi^{-1}(Y)$. If

$$-K_X.f_*C > \dim X + 1,$$

then

$$-K_X.f_*C - \dim X \geq 2.$$

Let $c' \in C$ be any point such that $f(c') \notin \pi^{-1}(y)$. By bend-and-break II (Proposition 6.6), $f_*C$ is equivalent to an effective connected reducible rational 1-cycle $Z$ which passes through $f(c)$ and $f(c')$. Thus, the irreducible components of $Z$ have $-K_X$-degree strictly smaller than $-K_X.f_*C$ and some component of $Z$ must meet $\pi^{-1}(y)$ but is not contained in $\pi^{-1}(y)$. Replace $f : C \to X$ with this component. Then we may repeat in this way until we arrive at a transverse rational curve with $-K_X.f_*C \leq \dim X + 1$.

This completes the proof in positive characteristic. Now we must lift to characteristic 0. Suppose $X$ is a Fano variety over an algebraically closed field $k$ of characteristic 0 and $\pi : X \to Y$. Since $X, Y$ and $\pi$ are of finite type, there exists some finitely generated $\mathbb{Z}$-algebra $R \subset k$, schemes $\mathscr{X} \to \operatorname{Spec} R$ and $\mathscr{Y} \to \operatorname{Spec} R$, a section $y_R : \operatorname{Spec} R \to \mathscr{Y}$, and a morphism $\pi_R : \mathscr{X} \to \mathscr{Y}$ of $R$-schemes such that

- $X = \mathscr{X} \otimes_R k$,
- $Y = \mathscr{Y} \otimes_R k$,

- $y_R \otimes_R k = y$, and
- $\pi = \pi_R \otimes_R k$.

Since $X/k$ and $Y/k$ are smooth and $\omega_{X/k}^{-1}$ is an ample line bundle, there exists a dense open subset $U \subset \operatorname{Spec} R$ such that $\mathscr{X}_U \to U$ and $\mathscr{Y}_U \to U$ are smooth and $\omega_{\mathscr{X}_U/U}^{-1}$ is relatively ample.

Let $\operatorname{Hom}_U^{\leq \dim X+1}(\mathbb{P}^1_U, \mathscr{X}_U)$ be the relative Hom scheme of families of rational curves on $\mathscr{X}_U$ of $\omega_{\mathscr{X}_U/U}^{-1}$-degree bounded by $\dim X + 1$. This is a quasi-projective scheme. There is the natural map

$$\rho_U : \operatorname{Hom}_U^{\leq \dim X+1}(\mathbb{P}^1_U, \mathscr{X}_U) \to \operatorname{Hom}_U(\mathbb{P}^1_U, \mathscr{Y}_U)$$

given by composing a family of rational curves with the map $\pi_U : \mathscr{X}_U \to \mathscr{Y}_U$. The scheme $\mathbb{P}^1_U$ comes equipped with two sections $0_U, \infty_U$. We consider the locally closed subscheme $\mathbb{H} \subset \operatorname{Hom}_U^{\leq \dim X+1}(\mathbb{P}^1_U, \mathscr{X}_U)$ given by the conditions

$$\rho_U(f)(0_U) = \pi_U^{-1}(y_U) \qquad \rho_U(f)(\infty_U) \cap y_U = \emptyset.$$

Then $\mathbb{H}$ is a quasi-projective $U$-scheme parametrizing rational curves of $\omega_{X_U/U}^{-1}$-degree bounded by $\dim X + 1$ which meet the fiber $\pi_U^{-1}(y_U)$ but are not contained in it. For every closed point $q \in U$, the residue field $k(q)$ has positive characteristic. Thus, by the desired theorem in positive characteristic, the fiber $\mathbb{H}_q$ is not empty. It follows that

$$\mathbb{H} \to U$$

is surjective onto closed points. Since $\mathbb{H}$ is quasi-projective, at least one component of $\mathbb{H}$ must dominate $U$. Therefore, the fiber over the generic point $k(U)$ is not empty but $k(U) \subset k$ so the base-change $\mathbb{H} \otimes_U k$ is also not empty. Since the Hom scheme commutes with base-change, $\mathbb{H} \otimes_U k$ is simply the scheme parametrizing rational curves on $X$ of $-K_X$-degree bounded by $\dim X + 1$ which meet $\pi^{-1}(y)$ but are not contained in $\pi^{-1}(y)$. Since $k$ is algebraically closed, this scheme has a $k$-point, and thus there exists such a rational curve on $X$.

$\square$

## 6.3 Fano varieties are rationally chain connected

We are finally ready to put together all the ingredients to prove that Fano varieties in characteristic 0 are rationally connected.

**Theorem 6.10.** *Let $k$ be an algebraically closed field and $X/k$ a smooth projective Fano variety. Then $X$ is rationally chain connected. If* $\operatorname{char}(k) = 0$*, then $X$ is rationally connected.*

*Proof.* If $\operatorname{char}(k) = 0$, then RCC implies RC by Theorem 5.6 which proves the latter claim assuming the former.

Let $\mu : X \dashrightarrow R(X)$ be the MRC fibration. If $\dim R(X) = 0$, then $X$ is rationally chain connected and we are done. Suppose towards a contradiction that $\dim R(X) > 0$. Then by Theorem 6.9, for any $z \in R(X)$ over which $\mu$ is well-defined and proper, there exists a rational curve $f : \mathbb{P}^1 \to X$ such $f(0) \in \mu^{-1}(z)$ but $f(\infty) \notin \mu^{-1}(z)$. On the other hand, by Theorem 5.13(3), for very general $z \in R(X)$, any rational curve which meets $\mu^{-1}(z)$ must be contained in $\mu^{-1}(z)$. This is a contradiction. $\square$

# 7 Birational superrigidity of the quartic threefold

In this section we will prove Theorem 1.26 of Iskovskikh and Manin – that a smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is not rational. Combined with the construction of B. Segre exhibiting a unirationl quartic threefold, this gives a counerexample to the Lüroth Problem.

The idea of Iskovskikh and Manin is to to show that the birational automorphism group $\mathrm{Bir}(X_4)$ is very small. In fact, something much stronger holds.

**Definition 7.1.** Let $X$ be a smooth Fano variety with Picard number $\rho(X) = 1$. We say $X$ is *birationally superrigid* if any birational map $\phi : X \dashrightarrow Y$ to another Fano variety with $\rho(Y) = 1$ is an isomorphism.

Note that a smooth hypersurface of dimension $\geq 3$ has Picard number 1 by the Lefschetz Hyperplane Theorem. Moreover, $X_4 \subset \mathbb{P}^4$, or more generally any $X_d \subset \mathbb{P}^{n+1}$ for $d \leq n+1$ is Fano.

**Theorem 7.2.** *(Iskovskikh-Manin [27]) A smooth quartic fourfold is birationally superrigid.*

The study of birational automorphism groups of Fano varieties was initiated by Noether and Fano and studied by many other people over the last century (e.g. [44, 12, 19] and many others) culminating in the following theorem [17, 18].

**Theorem 7.3.** *A smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of dimension $n \geq 3$ and degree $n+1$ is birationally superrigid.*

The full proof and history of Theorem 7.3 is outlined in [29]. In this section we will sketch only the proof of Theorem 7.2, following ideas of [16] as presented in [29].

*Remark* 7.4. Theorem 7.3 implies that $\mathrm{Aut}(X) = \mathrm{Bir}(X)$ for a hypersurface of degree $n+1$ in $\mathbb{P}^{n+1}$ with $n \geq 3$. To conclude Theorem 7.2, we need to know that $\mathrm{Aut}(X)$ is finite. To see this we first note that $\mathrm{Aut}(X)$ is a linear algebraic group. Indeed any automorphism preserves $-K_X$ but but $-K_X$ is very ample and thus every automorphism of $X$ extends to an automorphism of the ambient projective space in its anti-canonical embedding. Therefore, it suffices to show that the connected component of the identity $\mathrm{Aut}^0(X)$ is trivial. This follows from the computation

$$T_{\mathrm{Id}}\mathrm{Aut}(X) = H^0(X, T_X) = 0$$

which holds for any smooth hypersurface of degree $d \geq 3$ and dimension $n$ except for $(n, d) = (1, 3)$ (see e.g. [28, 11.5.2]).

## 7.1 Singularities of pairs

In this section we review some basic definition of singularities of pairs. For the proof of Theorem 7.2, it is convenient to work with a pair $(X, |M|)$ of a smooth projective variety $X$ equipped with a linear series $|M| \subset \mathbb{P}H^0(X, \mathscr{O}_X(M))$. We do not assume the linear series $|M|$ is complete. Moreover, we will denote a general member of $|M|$ by $M \in |M|$.

**Definition 7.5.** The *base locus* $\mathrm{Bs}|M|$ is defined as the intersection

$$\bigcap_{D \in |M|} D.$$

We say $|M|$ is *movable* if $\mathrm{Bs}|M|$ has codimension $\geq 2$. The *fixed part* of $|M|$ is the maximal effective divisor $F$ such that $D - F \geq 0$ for all $D \in |M|$.

Note that $\mathrm{Supp}F \subset \mathrm{Bs}|M|$ consists of exactly the codimension 1 components of $\mathrm{Bs}|M|$ so $|M|$ is movable if and only if the fixed part $F = 0$. Thus we see that any linear series can be written uniquely as

$$|M| = |M'| + F$$

where $|M'|$ is a movable linear series and $F$ is the fixed part of $|M|$.

**Definition 7.6.** Let $(X, |M|)$ be a pair. A *log resolution* of this tuple is a proper birational morphism $\mu : Y \to X$ such that

1. $Y$ is smooth,
2. $\mu^*|M| = |M'| + F$ where $|M'|$ is basepoint free and $F$ is the fixed part of $\mu^*|M|$, and
3. $F + \mathrm{Exc}(\mu)$ is a simple normal crossings divisor.

Given a log resolution $\mu : Y \to X$ of a pair $(X, |M|)$, if $X$ is smooth, we can write

$$K_Y = \mu^* K_X + E$$

where $E$ is effective and $\mu$-exceptional. Then for any rational number $c > 0$, we can write

$$K_Y + c|M'| = \mu^*(K_X + c|M|) + E - cF.$$

Let us denote

$$E - cF = \sum a_i E_i$$

where $E_i$ are prime divisors.

**Definition 7.7.** The pair $(X, c|M|)$ is *canonical (resp. log canonical)* if $a_i \geq 0$ (resp. $a_i \geq -1$). A divisor $E_i$ with $a_i < 0$ (resp $a_i < -1$) is called a *non-canonical (resp. non-log canonical) place* and the image $\mu(E_i) \subset X$ is a *non-canonical (resp. non-log canonical) center.*

One should think of a log resolution of a pair $(X, |M|)$ as resolving the base locus of the linear series $|M|$. Then the numbers $a_i$ are a measure of the singularities of the base locus and the conditions of being canonical or log canonical are conditions on the singularities of the base locus being mild. In particular if $|M|$ is basepoint free, then $(X, c|M|)$ is canonical for any $c$. In general, if $X$ is smooth, there exists a maximal $c$ such that $(X, c|M|)$ is canonical (resp. log canonical) called the canonical (resp. log canonical) threshold.

*Remark* 7.8. If $|M|$ is movable, then $F$ is necessarily $\mu$-exceptional so the non-canonical (resp. non-log canonical) centers must have codimension $\geq 2$.

*Remark* 7.9. It is not hard to see that $(X, c|M|)$ is (log) canonical if and only if $(X, cM)$ is (log) canonical where $M$ is a general member of $|M|$. Thus we see that the study if the singularities of the base locus of a linear series is essentially equivalent to the study of the singularities of the general member. This is consistent with the previous remark as the general member of a basepoint free linear series is smooth by Bertini's Theorem.

## 7.2 The Noether-Fano method

In this section, we will discuss the so-called Noether-Fano method for proving superrigidity (see [29, Section 2]. The basic idea is that the existence of a nontrivial birational map between Picard number 1 Fano varieties forces the existence of a movable linear series $|M|$ with bad singularities. Then a careful analysis of the singularities of such linear series on quartic threefolds will lead to Theorem 7.2.

**Theorem 7.10.** *Let $\phi : X \dashrightarrow Y$ be a birational map between smooth Fano varieties of Picard number 1. Then either*

1. *$\phi$ is an isomorphism, or*
2. *there is some $m > 0$ and a movable linear series $|M| \subset |-mK_X|$ such that $(X, \frac{1}{m}|M|)$ is not canonical.*

We will need the following lemma of Matsusaka and Mumford [37].

**Lemma 7.11.** *Let $\phi : X \dashrightarrow Y$ be a birational map between smooth projective varieties and let $H$ be an ample divisor on $X$. Suppose that $\phi_* H$ is ample and that there exist closed subsets $P \subset X$ and $Q \subset Y$ with complement of codimension 2 such that $\phi|_{X \setminus P} : X \setminus P \to Y \setminus Q$ is an isomorphism. Then $\phi$ is an isomorphism.*

*Proof.* Without loss of generality we may assume that $H$ and $\phi_* H$ are very ample on $X$ and $Y$ respectively. By the codimension condition, $\phi$ induces isomorphism

$$H^0(X, mH) = H^0(X \setminus P, mH|_{X\setminus P}) = H^0(Y \setminus Q, m\phi_* H|_{Y \setminus Q}) = H^0(Y, m\phi_* H)$$

for all $m \geq 0$ so $\phi$ is an isomorphism. $\square$

*Proof of Theorem 7.10.* Let $Z \to X \times Y$ be a log resolution of the closure of the graph of $\phi$ and consider the natural maps

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}.$$

Let $|N| \subset |-nK_Y|$ be a basepoint free linear series with strict transform $|M| = \phi_*^{-1}|N|$. Since $\rho(X) = 1$, $d|M| \subset |-mK_X|$ for some $m > 0$ and without loss of generality we may take $d = 1$.

We have an effective $p$-exceptional divisor $E_p$ (resp. $q$-exceptional divisor $E_q$) defined by

$$K_Z = p^* K_X + E_p \qquad (\text{resp. } K_Z = q^* K_X + E_q) \tag{13}$$

where $\mathrm{Supp}(E_p) = \mathrm{Exc}(p)$ (resp. $\mathrm{Supp}(E_q) = \mathrm{Exc}(q)$).

Moreover, if we let $|M_Z| = q^*|N|$, then $p^*|M| = |M_Z| + F_p$ where $F_p$ is also effective and $p$-exceptional by the negativity lemma [33, Lemma 3.39]. Note that $p_*|M_Z| = \phi_*^{-1}|N| = |M|$ by definition. Then

$$K_Z + \frac{1}{n}|M_Z| = q^*\left(K_Y + \frac{1}{n}|N|\right) + E_q \sim_{\mathbb{Q}} E_q \geq 0 \implies \tag{14}$$

$$K_X + \frac{1}{n}|M| = p_*\left(K_Z + \frac{1}{n}|M_Z|\right) \sim_{\mathbb{Q}} p_* E_q \geq 0. \tag{15}$$

On the other hand,

$$K_X + \frac{1}{n}|M| \sim_{\mathbb{Q}} \left(1 - \frac{m}{n}\right) K_X = \frac{m-n}{n}(-K_X) \geq 0 \tag{16}$$

so we conclude that $m \geq n$.

Similarly,

$$K_Z + \frac{1}{m}|M_Z| = p^* \left(K_X + \frac{1}{m}|M|\right) + E_p - \frac{1}{m}F_p \sim_{\mathbb{Q}} E_p - \frac{1}{m}F_p \implies \tag{17}$$

$$K_Y + \frac{1}{m}|N| = q_* \left(K_Z + \frac{1}{m_*}|M_Z|\right) \sim_{\mathbb{Q}} q_* \left(E_p - \frac{1}{m}F_p\right). \tag{18}$$

Then as before we obtain that

$$\frac{n-m}{m}(-K_Y) \sim_{\mathbb{Q}} q_* \left(E_p - \frac{1}{m}F_p\right). \tag{19}$$

If $n < m$, then the divisor 19 is not effective so by Equation 17, $\left(X, \frac{1}{m}|M|\right)$ is not canonical and we are in case (2) of the theorem. Otherwise, by Equation 16, we must have $n = m$. By Equations 14+3, $p_* E_q = 0$ so $E_q$ is $p$-exceptional but $E_q$ contains every $q$-exceptional divisor so every $q$-exceptional divisor if also $p$-exceptional. On the other hand, suppose $D \subset Z$ is not $q$-exceptional. Then $q_* D \sim_{\mathbb{Q}} r|N|$ for some $r > 0$ since $\rho(Y) = 1$. Pulling back, we see that

$$r|M_Z| \sim_{\mathbb{Q}} D + G$$

where $G$ is $q$-exceptional. But by the previous conclusion, this implies $G$ is also $p$-exceptional. Since $p_*|M_Z| = |M|$ is not 0, then $p_* D \neq 0$ so $D$ is not $p$-exceptional.

Putting this together, we have a rational map $\phi : X \dashrightarrow Y$ between Fano varieties such that

- $\phi_*(-K_X) = -K_Y$, and
- $\mathrm{Exc}(p) = \mathrm{Exc}(q)$.

Then $\phi$ is an isomorphism by Lemma 7.11 where we take $P = p(\mathrm{Exc}(p))$ and $Q = q(\mathrm{Exc}(q))$ and we are in case (1) of the theorem. □

**Corollary 7.12.** *Suppose $X$ is a smooth Fano variety with $\rho(X) = 1$. If $(X, \frac{1}{m}|M|)$ is canonical for every $m > 0$ and every movable linear series $|M| \subset |-mK_X|$, then $X$ is birationally superrigid.*

## 7.3 Multiplicity bounds

In this section we discuss multiplicity bounds that are used to show that a smooth quartic threefold satisfies the hypothesis of Corollary 7.12. On the one hand, Proposition 7.13 gives an upper bound on the multiplicity of $D \in |mH|$ where $H$ is the hyperplane class. On the other hand, Proposition 7.15 gives lower bounds on the multiplicity of a general member $M \in |M|$ for a non-canonical pair $(X, \frac{1}{m}|M|)$. The key point is that for the quartic, $-K_X = H$ so that the inear series $|M| \subset |-mK_X|$ produced by Theorem 7.10 are also degree $m$-hypersurface sections. Thus we can play off both types of multiplcity bounds to derive a contradiction.

We begin with the upper bound on the multiplicity of a hypersurface section $D \subset |mH|$.

**Proposition 7.13.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface with $n \geq 3$. Let $D \in |mH|$ where $H$ is a hyperplane section. Let $C \subset D \subset X$ be an integral curve. Then*

$$\mathrm{mult}_C D \leq m.$$

*Proof.* The proof uses results from intersection theory (see e.g. [22] for an introduction). First we compute the intersection number

$$D \cdot C = m \deg C$$

by Bezout's theorem. On the other hand, let $\Sigma \subset \mathbb{P}^{n+1}$ be the cone over $C$ with vertex $v$. The key point is that $\deg \Sigma = \deg C$. However, $X \cap \Sigma$ is not necessarily equal to $C$, instead we have a residual curve $Z$ with

$$X \cdot \Sigma = C + Z$$

**Claim 7.14.** *For $v$ general,*

$$\#Z \cap C = \deg C(\deg X - 1).$$

*Proof.* Let $\pi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ be the projection away from $v$. For general $v$, $\pi|_X : X \dashrightarrow \mathbb{P}^n$ is finite and $\pi|_C : C \to \mathbb{P}^n$ is an isomorphism onto its image. Moreover, we can pick $v$ so that $Z \cap C$ are smooth points of $C$. For any smooth point $x \in C$, denote by $l_{x,v} \subset \Sigma$ the line containing $x$ and $v$. Then $x \in Z \cap C$ if and only if $l_{x,v}$ meets $Z \cup C$ in multiplicity $\geq 2$. But $Z \cup C = \Sigma \cap X$ and $l_{x,v}$ is contained in $\Sigma$, this is equivalent to saying that $l_{x,v}$ is tangent to $X$ at $x$. The points $x \in X$ where $l_{x,v}$ is tangent to $X$ is exactly the ramification divisor of $\pi|_X$. Thus

$$Z \cap W = R \cap W$$

where $R$ is the ramification divisor of $\pi$. For $v$ generic, $R$ and $W$ meet transversely and one can compute explicitly that $\deg R = \deg X - 1$ so we conclude by Bezout. □

Now we compute

$$D \cdot C = D \cdot X \cdot \Sigma - D \cdot Z = m \deg X \deg C - D \cdot Z.$$

Finally we need the following fact about intersection multiplicity.

$$(D \cdot Z)_p \geq (\mathrm{mult}_p D)(\mathrm{mult}_p Z) \geq \mathrm{mult}_p D.$$

For $p \in C$, $\mathrm{mult}_p D \geq \mathrm{mult}_C D$ and we have

$$D \cdot Z \geq \mathrm{mult}_C W(\#Z \cap C) = \mathrm{mult}_C D \deg C(\deg X - 1)$$

Putting this together we get

$$m \deg C = D \cdot C = m \deg X \deg C - D \cdot Z \leq m \deg X \deg C - \mathrm{mult}_C D \deg C(\deg X - 1)$$

which, after rearranging, becomes $m(1 - \deg X) \leq \mathrm{mult}_C D(\deg X - 1)$ or $\mathrm{mult}_C D \leq m$.

□

Next we consider the lower bounds on multiplicity which are implied by the fact that the pair $(X, \frac{1}{m}|M|)$ is not (log) canonical.

**Proposition 7.15.** *Consider a pair $(X, c|M|)$ with $|M|$ movable and let $p \in X$.*

1. *If $(X, c|M|)$ is not canonical at $p$, then*

$$\mathrm{mult}_p M > 1/c.$$

2. *If $(X, c|M|)$ is not log canonical at $p$, then*

$$(M \cdot M)_p > 4/c^2$$

To prove Proposition 7.15, we need an important tool from the minimal model program which lets us compare the singularities of a pair $(X, S+\Delta)$ with the singularities of the pair $(S, \Delta|_S)$ where $S$ is a general hyperplane section and $\Delta$ is some $\mathbb{Q}$-divisor. The basic idea is that the Poincaré residue map, which is given in local coordinates around $S = \{x_1 = 0\}$ by

$$\mathrm{res} : \Omega_X(S) \to \omega_S$$
$$f(x_1, \ldots, x_n)\frac{dx_1}{x_1} \wedge dx_2 \wedge \ldots \wedge dx_n \mapsto f(0, x_2, \ldots, x_n)dx_2 \wedge \ldots \wedge dx_n$$

gives us the equality $(K_X + S)|_S = K_S$. By working on a log resolution of the pair $(X, S + \Delta)$ and restricting to the strict transform, its not hard to see that $(X, S + \Delta)$ log canonical in a neighborhood of $S$ implies that $(S, \Delta_S)$ is log canonical. The converse is an extremely powerful and nontrvial result that is useful for many inductive arguments in the minimal model program [33, Theorem 5.50].

**Theorem 7.16** (Inversion of adjunction)**.** *Let $X$ be a smooth projective variety, $\Delta$ an effective $\mathbb{Q}$-divisor and $S$ a smooth hypersurface. Then $(X, S + \Delta)$ is log canonical in a neighborhood of $S$ if and only if $(S, \Delta|_S)$ is log canonical.*

*Remark* 7.17. Note that for adjunction, $S$ must necessarily be a Cartier divisor, and in particular, must have coefficient 1. Thus the pair $(X, S + \Delta)$ is *never* canonical. Thus even if $(X, \Delta)$ is a canonical pair, to apply inductive arguments using inversion of adjunction we need to pick an appropriate $S$ such that $(X, S + \Delta)$ is log canonical and then relate this to the *log canonical* pair $(S, \Delta|_S)$.

*Remark* 7.18. Even though Theorem 7.16 is stated for a log pair $(X, S+\Delta)$, we can apply it to the setting of a linear series $c|M|$ by taking $\Delta$ to be a general member of $c|M|$ (Remark 7.9).

**Corollary 7.19.** *Let $(X, c|M|)$ be a pair and $S \subset X$ a smooth hypersurface. If either*

1. *$S$ contains a non-canonical center $Z^c$ of $(X, c|M|)$, or*

2. *$S$ intersects a non-log canonical $Z^{lc}$ center of $(X, c|M|)$,*

*then $(S, c|M|_S)$ is not log canonical at $p \in Z^c$ (resp. $p \in Z^{lc} \cap S$).*

*Proof.* For case (1), note that if $S$ contains a non-canonical center $Z^c$, then there exists some non-canonical place $E \subset Y$ lying over $Z^c$ on some resolution $\mu : Y \to X$. Then by assumption we have that the discrepancy $a(E) < 0$ where $a(E)$ is the coefficient of $E$ in the divisor $K_Y + c|M'| - \mu^*(K_X + c|M|)$. Since $S$ contains $Z^c = \mu(E)$, then $\mu^*S = S' + bE$ where $b > 0$ and $S'$ does not contain $\mathrm{Supp}(E)$. Since $S$ is Cartier, $b$ is an integer so $b \geq 1$. Thus the discrepancy of $E$ for the log pair $(X, S + c|M|)$ is $a - b \leq a - 1 < -1$ so $(X, S + c|M|)$ is not log canonical. By inversion of adjunction, this implies that $(S, c|M|_S)$ is not log canonical.

For case (2), $(X, c|M|)$ is already not log canonical so neither is $(X, S + c|M|)$ and we conclude again by inversion of adjunction. □

*Proof of Proposition 7.15.* For the first part, we consider a smooth complete intersection curve $p \in B \subset X$ through $p$, say $B = S_1 \cap \ldots \cap S_k$ where $S_i$ are smooth hypersurfaces through $p$. Suppose that $\mathrm{mult}_p M \leq 1/c$. Then $cM.B \leq 1$ thus $(B, cM|_B)$ is a log canonical curve. By applying Corollary 7.19 inductively to each $S_i$, we conclude that $S_i$ does not contain a non-canonical center in a neighborhood of $p$. In particular, $(X, c|M|)$ must be canonical at $p$.

For (2), the idea is to reduce to the case that $\dim X = 2$. Indeed if $S$ is a generic smooth hypersurface passing through $p$, then $(X, S + c|M|)$ is not log canonical at $p$ if and only if $(S, c|M|_S)$ is not log canonical at $p$ by inversion of adjunction. On the other hand, for generic $S$, the multiplicity of intersection is preserved,

$$(M \cdot M)_p = (M|_S \cdot M|_S)_p.$$

Thus it suffices to prove the statement for $S$ and so by induction we can slice down until $\dim X = 2$. For surfaces the statement can be proven directly by a delicate local computation in local coordinates $x$ and $y$ around $p$ (see [29, Corollary 31] for details).

□

## 7.4 Finishing the proof

We are now ready to prove that the smooth quartic threefold is birationally superrigid.

**Proposition 7.20.** *Let $X$ be a quartic threefold and let $|M| \subset |-mK_X|$ be a movable linear series. Then $\left(X, \frac{1}{m}|M|\right)$ is canonical.*

*Proof.* First note that for a quartic threefold, we have $-K_X = H$ where $H$ is the hyperplane class. Thus, $|M| \subset |mH|$.

Suppose that $\left(X, \frac{1}{m}|M|\right)$ is not canonical. Since $|M|$ is movable, its base locus has codim $\geq 2$ so $\left(X, \frac{1}{m}|M|\right)$ does not have any two dimensional non-canonical centers. By Proposition 7.15(1), $\mathrm{mult}_p M > m$ for all $p$ in a non-canonical center. On the other hand, for any curve $C \subset X$, $\mathrm{mult}_C M \leq m$ since $|M| \subset |mH|$ (Proposition 7.13). Thus, $\left(X, \frac{1}{m}|M|\right)$ does not contain any one dimensional non-canonical centers. Therefore, all the non-canonical centers of $\left(X, \frac{1}{m}|M|\right)$ are points.

Let $p \in X$ be a non-canonical center of $\left(X, \frac{1}{m}|M|\right)$ and let $S$ be a generic smooth hyperplane section passing through $p$. By Corollary 7.19, $\left(S, \frac{1}{m}|M|_S\right)$ is not log canonical at $p$.

By Proposition 7.15(2), $(M_S \cdot M_S)_p > 4m^2$. Since $S$ is generic,

$$(M \cdot M \cdot S)_p = (M_S \cdot M_S)_p > 4m^2.$$

On the other hand, $M \sim mH$ and $S \sim H$ and so

$$4m^2 = m^2 H^3 = M \cdot M \cdot S \geq (M \cdot M \cdot S)_p > 4m^2$$

which is a contradiction.

□

**Corollary 7.21.** *Let $X$ be a smooth quartic threefold. Then $X$ is birationally superrigid.*

## 7.5 Unirationality constructions and a counterexample to the Lüroth problem

In this section we will sketch the construction of B. Segre (see also [27, Section 9]) of a particular unirational quartic threefold. Combined with Theorem 7.2 we get the following.

**Corollary 7.22.** *There exists a unirational but irrational quartic threefold.*

Before we give the construction, let us state the following open problem.

**Question 7.23.** Is a general quartic threefold over an algebraically closed field unirational?

*Remark* 7.24. Note that by Theorem [25], we know that a smooth quartic in $\mathbb{P}^{n+1}$ is unirational for $n \gg 0$. In fact by work of Morin, the general smooth quartic is unirational for $n \geq 5$.

Let $p \in X$ and let $T_p \subset \mathbb{P}^4$ denote the projective tangent hyperplane to $X$ at $p$. The intersection $T_p \cap X$ is a quartic surface $S_p$ with a multiplicity $\geq 2$ singularity at $p$. For generic $p \in X$, we can assume that $S_p$ has a unique singular point of multiplicity exactly 2 at $p$. Let $C_p$ denote the tangent cone of $S_p$ at $p$. Then $C_p$ is a degree 2 cone since $p$ is a multiplicity 2 point of $S_p$. For generic $p$, we have that $C_p$ is integral. Thus $C_p$ is the cone over conic curve. The lines through the cone $C_p$ are exactly the lines which are tangent to $X$ at $p$ in multiplicity at least 3.

Let $W \subset \mathbb{P}TX \to X$ be the subvariety of the projectivized tangent bundle consisting of those tangent directions which are tangent to $X$ at multiplicity at least (3). Then by the previous discussion, for generic $p \in X$, the fiber $W_p$ is the conic so that $C_p = \operatorname{cone}(W_p)$. Thus $\pi : W \to X$ is a conic bundle. On the other hand, there is a map

$$f : W \dashrightarrow X$$

given by the fourth point construction. Each tangent direction at $w \in W$ corresponds to a tritangent line at $x$. Since $\deg X = 4$, then this line intersects $X$ in a unique second point which we call $f(w)$.

We want to use $f$ to produce a unirational parametrization. The issue here is that $W$ need not be rational, it is merely a conic bundle over $X$. However, if $S$ contains a (possibly singular) rational surface $S \subset X$, then we could hope that the pullback $\pi_S : W_S \to S$ has a section. If this is the case, then $W_S$ is rational by Proposition 3.7. We could also hope that in this case the fourth point map $f_S : W_S \dashrightarrow X$ is still dominant. In this case, $f_S : W_S \dashrightarrow X$ is a rational parametrization of $X$.

By beginning with a rational surface $S$ such that $f_S$ has a section and then finding a smooth quartic that contains in, B. Segre was able to find an explicit example of a quartic threefold where this strategy succeeds in finding a unirational parametrization. The quartic is given by the equation below.

$$x_0^4 + x_0 x_4^3 + x_1^4 - 6x_1^2 x_2^2 + x_2^4 + x_3^4 + x_3^3 x_4 = 0$$

# 8 The intermediate Jacobian of a cubic threefold

In this section we will survey the work of Clemens-Griffiths [13] on the intermediate Jacobian $\mathrm{IJ}(X)$ of a cubic threefold. Our goal will be to sketch the proof of the following theorem.

**Theorem 8.1.** *Let $X$ be a smooth projective threefold with $h^{1,0}(X) = h^{3,0}(X) = 0$. If $X$ is rational, then $\mathrm{IJ}(X)$ is isomorphic to a product of Jacobians. As a result we have the following.*

Via a careful analysis of the singularities of the theta divisor of $\mathrm{IJ}(X)$, Clemens and Griffiths conclude that when $X \subset \mathbb{P}^4$ is a smooth cubic threefold, then $\mathrm{IJ}(X)$ is *not* isomorphic to a product of Jacobians.

**Theorem 8.2.** *Smooth cubic threefolds over $\mathbb{C}$ are not rational.*

*Remark* 8.3. In this section, we work over $\mathbb{C}$. Benoist-Wittenberg extended the Clemens-Griffiths method to non-closed fields in [6].

## 8.1 Background on abelian varieties

Let $V$ be a complex vector space.

**Definition 8.4.** A *lattice* $\Gamma \subset V$ is a free abelian subgroup of $V$ such that $\mathrm{Span}_{\mathbb{R}}\Gamma = V$. The associated *complex torus* is the complex manifold $A = V/\Gamma$. We say that $A$ is an *abelian variety* if $A$ is the complex manifold of $\mathbb{C}$-points of a projective variety.

In particular, if $\Gamma \subset V$ is a lattice, then $\mathrm{rank}_{\mathbb{Z}}\Gamma = 2\dim_{\mathbb{C}} V$ and $A$ is homeomorphic to the torus $(S^1)^{2\dim_{\mathbb{C}} V}$. The lattice $\Gamma \subset V$ can be recovered as the integral homology group $H_1(A, \mathbb{Z})$ where $V = H^0(A, \Omega^1_A)^\vee$. The inclusion

$$H_1(A, \mathbb{Z}) \hookrightarrow H^0(A, \Omega^1_A)^\vee$$

is given by

$$\gamma \mapsto \left(\omega \mapsto \int_\gamma \omega\right). \tag{20}$$

**Definition 8.5.** A *polarization* is a non-degenerate skew-symmetric bilinear form

$$Q : \Gamma \times \Gamma \to \mathbb{Z}$$

such that

1. the extension $Q_{\mathbb{R}} : V \times V \to \mathbb{R}$ satisfies $Q_{\mathbb{R}}(iv, iw) = Q_{\mathbb{R}}(v, w)$, and
2. the Hermitian form
$$H(v, w) = Q_{\mathbb{R}}(iv, w) + iQ_{\mathbb{R}}(v, w)$$
is positive definite.

The polarization is *principal* if $Q$ is unimodular.

*Remark* 8.6. Condition (1) on $Q_{\mathbb{R}}$ gurantees that $H$ is indeed Hermitian. Note that $H(v, v) \in \mathbb{R}$ for a Hermitian form so it makes sense to ask for $H$ to be positive definite.

The bilinear form $Q$ induces a homomorphism $\Gamma \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) = \Gamma^\vee$ which then induces a map

$$\phi : A \to A^\vee := V^\vee/\Gamma^\vee.$$

In particular, $Q$ is principal if and only if $\phi$ is an isomorphism. On the other hand, we can identify

$$\mathrm{Hom}(\Lambda^2 H_1(A, \mathbb{Z}), \mathbb{Z}) = H^2(A, \mathbb{Z})$$

by Poincaré duality. Under this identification, $Q$ corresponds to a divisor class $\theta$ well defined up to numerical equivalence. The fact that $H$ is positive definite corresponds to the property that $\theta$ is ample. Finally, if $L$ is a choice of ample line bundle with $c_1(L) = \theta$ inducing the morphism $\phi$, then the degree $\deg \phi$ is given by $\chi(L)^2$. In particular, $\phi$ is an isomorphism (i.e. $Q$ is principal) if and only if $H^0(A, L) = 1$.[3] This implies that for a principal polarization, the divisor class $\theta$ gives a well defined divisor, also denoted by $\theta$, up to translation by $A$.

**Definition 8.7.** A *principally polarized abelian variety* (ppav) is a pair $(A, \theta)$ of a complex torus $A$ equipped with a principal polarization $\theta$.

*Remark* 8.8. Note that a complex torus $A$ is projective, that is, $A$ is an abelian variety, if and only if it admits a polarization.

Let $X$ be a smooth projective variety.

**Definition 8.9.** The *Picard variety* $\mathrm{Pic}^0(X)$ is the space of line bundles on $X$ algebraically equivalent to $\mathscr{O}_X$. It is the connected component of the identity in the Picard scheme $\mathrm{Pic}(X)$.

In terms of linear algebra, $\mathrm{Pic}^0(X)$ is given by $V = H^1(X, \mathscr{O}_X)$ and $\Gamma = H^1(X, \mathbb{Z})$ as can be seen by taking cohomology of the exponential sequence[4]

$$0 \to Z \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0.$$

Given an ample line bundle $L$ on $X$, we obtain a polarization on $\mathrm{Jac}(X)$ by

$$H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z} \qquad (\gamma_1, \gamma_2) \mapsto \int_X \gamma_1 \cup \gamma_2 \cup c_1(L)^{\dim X - 1}.$$

Note that this polarization is not necessarily principal.

**Proposition 8.10.** *Let $A$ be an abelian variety. Then the the picard variety* $\mathrm{Pic}^0(A)$ *is isomorphic to the dual abelian variety $A^\vee$.*

*Proof.* Granted the existence of $\mathrm{Pic}^0(A)$ and the claim that it is described by the lattice $H^1(A, \mathbb{Z})$, then this boils down to the duality between $H^1(A, \mathbb{Z})$ and $H_1(A, \mathbb{Z})$. □

The power of the above proposition is that we can use it to define many maps between $A$ and $A^\vee$. Indeed, for an element $a \in A$, let us denote the translation by

$$t_a : A \to A \qquad t_a(x) = x + a.$$

Then given any divisor $D$ on $A$, the difference $t_a^* D - D$ is algebraically equivalent to 0. Thus we can define the map

$$\varphi_D : A \to A^\vee \qquad \varphi_D(a) = \mathscr{O}_A(t_a^* D - D) \tag{21}$$

For a general $X$, the Picard variety is dual to the Albanese variety. Let $V = H^0(X, \Omega_X^1)^\vee$ and $\Gamma = H_1(X, \mathbb{Z})/\mathrm{tors}$ where the map $\Gamma \to V$ is as in Equation 20.

[3]Here we are using that $H^i(A, L) = 0$ for $i > 0$ by e.g. Kodaira Vanishing so $\chi(L) = H^0(A, L)$.

[4]Note that $H^(X, \mathbb{Z})$ is always torsion free

**Definition 8.11.** The *Albanese variety* of $X$ is an abelian variety $\mathrm{Alb}(X)$ associated to the data $H_1(X, \mathbb{Z})/\mathrm{tors} \subset H^0(X, \Omega^1_X)^\vee$.

$\mathrm{Alb}(X)$ is characterized by the following universal property. For any $x_0 \in X$, there exists a map

$$AJ_{x_0} : X \to \mathrm{Alb}(X) \qquad x_0 \mapsto 0$$

where 0 is the identity element. Moreover, $AJ_{x_0}$ is universal for maps $X \to A$ with $x_0 \mapsto 0_A$.

We can see the duality between $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$ as follows. Given any map $X \to A$, we have a pullback $\mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$, dualizing and using the fact that $\mathrm{Pic}^0(A) = A^\vee$, we obtain a map $\mathrm{Pic}^0(X)^\vee \to A$. Unraveling the definitions shows that this is the map $\mathrm{Alb}(X) \to A$ guaranteed by the universal property.

The category of principally polarized abelian varieties (ppavs) is particularly well behaved. In particular, it has unique decomposition into irreducibles.

**Definition 8.12.** A ppav $(A, \theta)$ is *reducible* if there exist nontrivial ppavs $(A_1, \theta_1)$ and $(A_2, \theta_2)$ such that $(A, \theta) \cong (A_1 \times A_2, p_1^*\theta_1 + p_2^*\theta_2)$. A ppav is *irreducible* if it is not reducible.

It is clear from the linear algebraic point of view that the product $(A_1 \times A_2, p_1^*\theta_1 + p_2^*\theta_2)$ is principally polarized. Indeed, a direct sum of unimodular lattices is unimodular. On the other hand, the theta divisor of a ppav is unique up to translation, and thus any divisor in the numerical class of $p_1^*\theta_1 + p_2^*\theta_2$ splits as a sum of pullbacks of theta divisors on $A_1$ and $A_2$ so the notion of reducibility is well defined. Moreover, if $(A, \theta)$ is reducible, then $\theta$ is a reducible divisor.

**Lemma 8.13.** *A ppav $(A, \theta)$ is irreducible if and only if the $\theta$ is irreducible. Moreover, each ppav admits a unique decomposition into a product of irreducible ppavs.*

*Proof.* We proved one direction above. On the other hand, suppose that $\theta = \theta^1 + \theta^2 + \ldots + \theta^n$. Each component $\theta^i$ defines a degenerate pairing $Q_i : \Gamma \times \Gamma \to \mathbb{Z}$ and thus a homomorphism $\varphi_i : A \to A^\vee$. Equivalently, this is the homomorphism $\varphi_{\theta_i}$ as in Equation 21. Then the image of $A_i^\vee = \mathrm{im}(\varphi_i)$ is an abelian subvariety of $A^\vee$ and its preimage $A_i = \varphi^{-1}(\mathrm{im}(\varphi_i))$ under the map $\varphi : A \to A^\vee$ induced by $\theta$. Then we take $\theta_i := \theta^i|_{A_i}$ and check that

$$(A, \theta) \cong (A_1, \theta_1) \times \ldots \times (A_n, \theta_n).$$

Uniqueness follows from the fact that $H^0(A, \mathscr{O}_A(\theta)) = 1$. $\square$

## 8.2 Jacobians of curves

Let $C$ be a smooth projective curve of genus $g$. Then the intersection product induces a unimodular pairing

$$H_1(C) \times H_1(C) \to \mathbb{Z}$$

induces a principal polarization on $\mathrm{Alb}(C)$. Thus we have map

$$\sigma : \mathrm{Alb}(C) \to \mathrm{Pic}^0(C).$$

Fixing a basepoint $x_0 \in C$, then we have a commutative diagram

$$\begin{tikzcd} & \mathrm{Alb}(C) \arrow[dd, "\sigma"] \\ C \arrow[ur, "AJ_{x_0}"] \arrow[dr, "\phi"'] & \\ & \mathrm{Pic}^0(C) \end{tikzcd}$$

where $\phi(x) = \mathscr{O}_C(x - x_0)$.

**Theorem 8.14** (Abel-Jacobi)**.** *The map $\sigma$ is an isomorphism and the diagram commutes.*

Since $\sigma$ is indepdent of choice of line bundle, it gives a canonical identification between $\mathrm{Alb}(C)$ and $\mathrm{Pic}^0(C)$ for curves. We call this ppav the *Jacobian* $\mathrm{Jac}(C)$ and denote the theta divisor by $\theta_C$.

There are also higher Abel-Jacobi maps

$$AJ^k_{x_0} : \mathrm{Sym}^k(C) \to \mathrm{Jac}(C) \qquad \sum_{i=1}^k x_i \mapsto \mathscr{O}_C\left(\sum_{i=1}^k x_i - kx_0\right).$$

If we don't twist down by $kx_0$, we obtain an unnormalized Abel-Jacobi map

$$AJ^k : \mathrm{Sym}^k(C) \to \mathrm{Pic}^k(C) \qquad \sum x_i \mapsto \mathscr{O}_C\left(\sum x_i\right)$$

to the component of $\mathrm{Pic}(C)$ of degree $k$ line bundles.

**Definition 8.15.** Let $W^r_k \subset \mathrm{Pic}^k(C)$ denote the subvariety

$$\{L \in \mathrm{Pic}^k(C) \mid h^0(C, L) \geq r + 1\}.$$

The image of the unnormalized Abel-Jacobi map $AJ^k$ is $W^0_k$. Since translating by $kx_0$ is an isomorphism, we can identify $W^0_k \subset \mathrm{Pic}^k(C)$ with the image of the normalized Abel-Jacobi map $AJ^k_{x_0}$ inside $\mathrm{Jac}(C)$.

**Theorem 8.16.** *The homology class of the image of $AJ^k_{x_0}$ is given by*

$$[AJ^k_{x_0}(\mathrm{Sym}^k(C))] = \frac{\theta^{g-k}}{(g-k)!} \in H_{2k}(\mathrm{Jac}(C), \mathbb{Z})$$

In particular, the class $\frac{\theta^{g-k}}{(g-k)!}$ is the class of an algebraic subvariety. This motivates the following definition.

**Definition 8.17.** Given a ppav $(A, \theta)$ with $\dim A = g$, we define the minimal class

$$\frac{\theta^{g-k}}{(g-k)!} \in H_{2k}(A, \mathbb{Z}).$$

It is not obvious, but true, that the minimal class is indeed an inegral class. While $\theta^{g-k}$ is always algebraic, and in fact the class of a subvariety, the minimal class need not be.

**Definition 8.18.** We say $(A, \theta)$ has *level $k$* if

$$\frac{\theta^{g-k}}{(g-k)!}$$

is an effective algebraic class.

By Theorem 8.16, the minimal class of a Jacobian of a curve is effective algebraic for any $k$ so $(\mathrm{Jac}(C), \theta_C)$ has level $k$ for all $k \geq 1$. Note that every ppav of dimension $g$ has level $g-1$ since $\theta$ is the class of a divisor.

**Definition 8.19.** Let $(A, \theta)$ be a ppav, we denote by $(A, \theta)_1$ the product of irreducible components of $(A, \theta)$ which are *not* level 1.

The significance of this definition is made clear by the following theorem of Matsusaka.

**Theorem 8.20** (Matsusaka)**.** *Let $(A, \theta)$ be an irreducible ppav. Then $(A, \theta)$ is a Jacobian if and only if it has level* 1*.*

Thus $(A, \theta)_1$ is the product of components of $(A, \theta)$ which are not isomorphic to Jacobians.

*Remark* 8.21. Note that $(\mathrm{Jac}(C), \theta_C)$ is irreducible since $\theta_C \cong AJ^{g-1}(\mathrm{Sym}^{g-1}(C))$ is irreducible.

**Theorem 8.22** (Riemann Singularity theorem)**.** *Let $(\mathrm{Jac}(C), \theta_C)$ be the Jacobian of a curve $C$. Under the isomorphism $\theta_C \cong W_{g-1}$, the singularities of $\theta_C$ are determined by the equality*

$$\mathrm{mult}_{[L]} W^0_{g-1} = H^0(C, L).$$

In particular, the singular locus of $\theta_C$ can be identified with the *Brill-Noether variety*

$$W^1_{g-1} = \{L \mid h^0(C, L) \geq 2\} \subset W^0_{g-1} \subset \mathrm{Pic}^{g-1}(C).$$

Using some Brill-Noether theory, one can compute the dimension of $W^1_{g-1}$ for any smooth curve $C$.

**Theorem 8.23.** *Let $C$ be a smooth projective curve. Then the dimension of the singular locus of $\theta_C$ is $g-4$ if $C$ is not hyperelliptic and $g-3$ if $C$ is hyperelliptic.*

These results can be used to prove the Torelli theorem and form the basis of the Andreotti-Mayer approach to the *Schottky problem* of distinguishing Jacobians among all ppavs. For us, the point is that this gives us an easy criterion to check that an irreducible ppav is *not* a Jacobian, namely, if the singular locus of $\theta$ is too small. Note that a generic ppav should have smooth theta divisor which shows that a generic ppav is not a Jacobian for $g \geq 4$.

## 8.3 The intermediate Jacobian

Let $X$ be a smooth projective 3-fold with $h^{3,0}(X) = h^{1,0}(X) = 0$. Note in particular, rationally connected $X$ satisfy this by Proposition 5.3. Since our goal is to obstruct rationality of $X$, we may as well assume that $X$ is rationally connected.

By Hodge theory, $H^3(X, C) = H^3(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{2,1} \oplus H^{1,2}$ with $H^{2,1} \cong \overline{H^{1,2}}$ the complex conjugate. This implies that the composition

$$H^3(X, \mathbb{Z})/\mathrm{tors} \to H^3(X, \mathbb{C}) \to H^{1,2}(X)$$

embeds $H^3(X, \mathbb{Z})/\mathrm{tors}$ as a lattice $\Gamma \subset V = H^{1,2}(X)$.

**Definition 8.24.** The *intermediate Jacobian* $\mathrm{IJ}(X)$ of a rationally connected threefold $X$ is the complex torus $V/\Gamma$ where

$$V = H^{1,2}(X) = H^2(X, \Omega^1_X) \qquad \Gamma = H^3(X, \mathbb{Z})/\mathrm{tors}.$$

In fact we can equip $\mathrm{IJ}(X)$ with a polarization as follows. Let

$$Q : \Gamma \times \Gamma \to \mathbb{Z} \qquad (\gamma_1, \gamma_2) \mapsto \int_X \gamma_1 \cup \gamma_2$$

denote the intersection pairing on $H^3(X, \mathbb{Z})$. This is a unimodular pairing by Poincaré duality. To produce a polarization, we need to modify this pairing a bit. We consider the pairing

$$H : V \times V \to \mathbb{C} \qquad (\alpha, \beta) \mapsto 2i \int_X \alpha \wedge \bar{\beta}.$$

In general, the *Hodge-Riemann bilinear relations* guarantee that $H$ is a positive definite Hermitian pairing on the *primitive cohomology*

$$H^3_{prim}(X) := \ker\left(\cdot c_1(L) : H^3(X, \mathbb{Z}) \to H^5(X, \mathbb{Z})\right)$$

where $L$ is a fixed ample line bundle on $X$ and $c_1(L) \in H^{1,1}(X)$ is its first Chern class viewed as a $(1,1)$-form. Under the assumption that $h^{1,0} = h^{2,3} = 0$[5], the map

$$\cdot c_1(L) : H^{1,2}(X) \to H^{2,3}(X) = 0$$

is the zero map. Thus all of $H^{1,2}(X)$ is primitive and $H$ is positive definite. Moreover, the imaginary part of $H$ is the intersection pairing $Q$ by definition. Since $Q$ is unimodular, this equips $\mathrm{IJ}(X)$ with a principal polarization $\theta_X$.

*Remark* 8.25. The intermediate Jacobian can be defined more generally in higher dimension and without any vanishing conditions using the Hodge filtration on odd dimensional cohomology. However, it generally will not admit a polarization.

In order to use the Hodge theoretic invariant $(\mathrm{IJ}(X), \theta_X)$ to study the birational geometry of $X$, we need to understand how $H^3(X)$ behaves under birational transformations. More generally, we have the following.

**Lemma 8.26.** *Let $X$ be a smooth projective variety and $Z \subset X$ a smooth closed subvariety of codimension $c$. Let $p : Y \to X$ be the blowup of $X$ along $Z$. Then there is an isomorphism*

$$H^p(Y, \mathbb{Z}) \cong H^p(X, \mathbb{Z}) \bigoplus \sum_{k=1}^{c-1} H^{p-2k}(Z, \mathbb{Z})$$

*which is compatible with Hodge structures.*

*Proof.* Let $\pi : E \to Z$ denote the exceptional divisor. For a blowup, $E \cong \mathbb{P}(N_{Z/X})$ and thus comes equipped with a relative $\mathscr{O}_E(1)$. Let $h = c_1(\mathscr{O}_E(1))$.

We can construct the map from the right to the left by constructing a map on each summand. The map $H^p(X, \mathbb{Z}) \to H^p(Y, \mathbb{Z})$ is simply pullback. For $H^{p-2k}(Z, \mathbb{Z}) \to H^p(Y, \mathbb{Z})$ we consider the composition

$$H^{p-2k}(Z, \mathbb{Z}) \xrightarrow{\pi^*} H^{p-2k}(E, \mathbb{Z}) \xrightarrow{\cdot h^{k-1}} H^{p-2}(E, \mathbb{Z}) \xrightarrow{j_*} H^p(Y, \mathbb{Z})$$

[5]That $h^{1,0} = h^{2,3}$ follows from Serre duality.

where $\cdot h^l$ denotes intersecting with the relatively ample class $l$ times and $j : E \hookrightarrow Y$ is the inclusion. Each of these maps is a map of Hodge structures so we just need to show that the sum is an isomorphism of vector spaces. Let $U = X \setminus Z = Y \setminus E$. Then we can consider the long exact sequence of pairs for $(X, U)$ and $(Y, U)$ respectively. Then $p$ is a map of pairs so pullback $p^*$ gives a map of long exact sequences of pairs. The key point now is that by excision and the Thom isomorphism, $H^{k-1}(X, U) = H^{k-2c}(Z)$ and $H^{k-1}(Y, U) = H^{k-2}(E)$ for any $k$. On the other hand, the cohomology of $E$ is described by the projective bundle formula.

**Lemma 8.27.** *Let $E \to Z$ be a projective bundle associated to a vector bundle $V \to Z$ of rank $r$. Then pullback induces an isomorphism of graded rings*

$$H^*(E, \mathbb{Z}) \cong H^*(Z, \mathbb{Z})[h]/(h^r + c_1 h^{r-1} + c_2 h^{r-2} + \ldots + c_r)$$

*where $c_i = c_i(V)$ are the Chern classes of $V$.*

□

**Corollary 8.28.** *If $\dim X = 3$ and $Z = C$ is a smooth curve, then*

$$H^3(Y, \mathbb{Z}) = H^3(X, \mathbb{Z}) \oplus H^1(C, \mathbb{Z}).$$

*In particular, $Y \to X$ is the blowup of a smooth threefold along a smooth curve, then*

$$(\mathrm{IJ}(Y), \theta_Y) \cong (\mathrm{IJ}(X), \theta_X) \times (\mathrm{Jac}(C), \theta_C)$$

Putting all this together, we have the first main theorem of [13].

**Theorem 8.29** (Clemens-Griffiths Criterion)**.** *Let $X$ be a smooth projective threefold. Suppose $X$ is rational. Then $\mathrm{IJ}(X)$ is isomorphic as a ppav to a product of Jacobians of curves. More generally, if $X$ and $Y$ are birational smooth projective threefolds with $h^{3,0} = h^{1,0} = 0$, then there is an isomorphism $(\mathrm{IJ}(X), \theta_X)_1 \cong (\mathrm{IJ}(Y), \theta_Y)_1$ between the level $\neq 1$ components of the intermediate Jacobian.*

*Proof.* Suppose $Y \dashrightarrow X$ is a birational map between threefolds with $h^{3,0} = h^{1,0} = 0$. Note that this vanishing is a birational invariant by Proposition 2.5 so if one of $X$ or $Y$ satisfies it, so does the other. Then we can resolve singularities of the map by blowing up $Y$ along smooth centers to obtain a diagram

$$\begin{tikzcd} & W \arrow[dl, "b"'] \arrow[dr, "f"] & \\ Y \arrow[rr, dashed] & & X \end{tikzcd}$$

where $b$ is proper and birational and $f$ is a composition of blowups at smooth centers. By Lemma 8.26 and Corollary 8.28, the blowup at points does not change the intermediate Jacobian while the blowup along a curve introduces a Jacobian factor to the intermediate Jacobian. Thus,

$$(\mathrm{IJ}(W), \theta_W) \cong (\mathrm{IJ}(Y), \theta_Y) \times \prod_i (\mathrm{Jac}(C_i), \theta_{C_i})$$

where $C_i$ are the sequence of curves that are blown up to produce $b$. Thus we have an isomorphism

$$(\mathrm{IJ}(W), \theta_W)_1 \cong (\mathrm{IJ}(Y), \theta_Y)_1.$$

On the other hand, $f$ is a proper and birational morphism so we have a well defined $f_*$ such that $f_* f^* = 1$. Therefore, $f^*$ embeds $(\mathrm{IJ}(X), \theta_X)_1$ as a summand of $(\mathrm{IJ}(W), \theta_W)_1 \cong (\mathrm{IJ}(Y), \theta_Y)_1$. Running the argument in reverse, i.e. picking a $W'$ which is a blowup of $X$ at smooth centers and admits a proper birational morphism to $Y$, we also get that $(\mathrm{IJ}(Y), \theta_Y)_1$ embeds as a direct summand of $(\mathrm{IJ}(X), \theta_X)_1$. By uniquness of the decomposition of a ppav into irreducible summands, we must have that

$$(\mathrm{IJ}(X), \theta_X)_1 \cong (\mathrm{IJ}(Y), \theta_Y)_1.$$

In the special case that $X$ is rational, then $Y = \mathbb{P}^3$ which has no odd cohomology so $(\mathrm{IJ}(X), \theta_X)_1 \cong (\mathrm{IJ}(Y), \theta_Y)_1 = 0$ and $\mathrm{IJ}(X)$ is a product of Jacobians of curves. $\square$

## 8.4 The cubic threefold

Now we specialize the previous discussion to the case of the cubic threefold $X_3 \subset \mathbb{P}^4$.

First let us determine the Hodge numbers of $X$. Since $X$ is rationally connected, we know that $h^{i,0} = h^{0,i} = 0$ by Theorem 6.3 and Proposition 5.3. Serre duality tells us that $h^{i,j} = h^{3-i,3-j}$. Finally, $h^{3,3} = h^{0,0} = 1$ and $h^{1,1} = \rho(X) = 1$ by the Lefschetz Hyperplane Theorem.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & & h^{2,1} & & h^{1,2} & & 0 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

**Lemma 8.30.** *Let $X$ be a smooth cubic threefold. Then*

$$h^{1,2} = h^{2,1} = h^2(X, \Omega_X^1) = 5.$$

*Proof.* Note that by adjunction, $\omega_X \cong \omega_{\mathbb{P}^4}(3)|_X = \mathscr{O}_X(-2)$. Consider the cotangent sequence

$$0 \to \mathscr{O}_X(-3) \to \Omega_{\mathbb{P}^4}|_X \to \Omega_X \to 0$$

where $\mathscr{O}_X(-3) = \mathscr{I}_X/\mathscr{I}_X^2$. By Serre duality and Kodaira vanishing,

$$\begin{aligned} h^2(X, \mathscr{O}_X(-3)) = h^1(X, \mathscr{O}_X(1)) &= h^1(X, \omega_X(3)) = 0 \\ h^3(X, \mathscr{O}_X(-3)) &= h^0(X, \mathscr{O}_X(1)). \end{aligned}$$

Using the ideal sequence

$$0 \to \mathscr{O}_{\mathbb{P}^4}(-3) \to \mathscr{O}_{\mathbb{P}^4} \to \mathscr{O}_X \to 0$$

we can compute $h^0(X, \mathscr{O}_X(1)) = h^0(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(1)) = 5$. The Euler sequence gives us

$$0 \to \Omega_{\mathbb{P}^4}|_X \to \mathscr{O}_X(-1)^{\oplus 5} \to \mathscr{O}_X \to 0.$$

Now $h^i(X, \mathscr{O}_X(-1)) = h^i(X, \omega_X(1)) = 0$ for all $i > 0$ by Kodaira vanishing. Moreover, $h^i(X, \mathscr{O}_X) = 0$ for all $i > 0$ by Proposition 5.3. Thus,

$$h^i(X, \Omega_{\mathbb{P}^4}|_X) = 0$$

for $i > 1$. In particular,

$$h^2(X, \Omega_X^1) = h^3(X, \mathscr{O}_X(-3)) = h^0(X, \mathscr{O}_X(1)) = 5.$$

$\square$

**Corollary 8.31.** *The intermediate Jacobian* $(\mathrm{IJ}(X), \theta_X)$ *of a smooth cubic threefold* $X_3 \subset \mathbb{P}^4$ *is ppav of dimension* 5.

The goal now is to study the geometry of $(\mathrm{IJ}(X), \theta_X)$ in order to show that it cannot be a Jacobian of a genus 5 curve. Recall that to access the geometry of the Jacobian and its theta divisor, we used the Abel-Jacobi maps $AJ^k_{x_0} : \mathrm{Sym}^k(C) \to \mathrm{Jac}(C)$. The analogue for the cubic threefold comes from studying the Fano variety of lines. Recall that by Theorem 3.12, the Fano variety of lines $F(X, 1)$ on a cubic threefold is a smooth surface which we will denote by $S$.

### 8.4.1 The Fano surface of lines on a cubic threefold

In order to study the geometry of $S$, we can make use of the following geometric construction. First, we define $W$ as the incidence variety

$$W = \{(x, L) \mid x \in L\} \subset X \times \mathbb{G}\mathrm{r}(1, 4).$$

Given two distinct points $y, z \in X$, we obtain a line $L_{y,z}$. If $y, z$ are general, then $L_{y,z}$ intersects $X$ in a unique third point $x$. In this way we get a rational map

$$t : \mathrm{Sym}^2(X) \dashrightarrow W \qquad \{y, z\} \mapsto (x, L_{y,z}).$$

Now $\mathrm{Sym}^2(X)$ is singular but it can be resolved by a single blowup

$$\mathrm{Bl}_\Delta \mathrm{Sym}^2(X) \to \mathrm{Sym}^2(X)$$

where $\Delta$ is the diagonal consisting of pairs $\{x, x\}$. In fact, a point of the blowup above $\{x, x\}$ corresponds to a tangent direction at $x \in X$. A tangent direction $v \in \mathbb{P}T_{X,x}$ can be identified with the length two subscheme $\Sigma$ with

$$\mathscr{I}_\Sigma = \{f \in \mathscr{O}_X \mid f(x) = 0,\ \ df(v) = 0\} \subset \mathscr{O}_X$$

and conversely, every length two subscheme of $X$ supported at $x$ is of this form. Thus,

$$\mathrm{Bl}_\Delta \mathrm{Sym}^2(X) \cong \mathrm{Hilb}^2(X)$$

the Hilbert scheme of two points on $X$.

Any nonreduced length two subscheme determines a line $L$, namely, the line through $x$ in the direction of $v$. Thus rational map $t$ extends to $\mathrm{Hilb}^2(X)$ and is well defined at any length two subscheme $\Sigma$ as long as the linear span of $\Sigma$ is not contained in $X$.

Let $U \subset \mathrm{Hilb}^2(X)$ be the locus of $\Sigma$ such that the linear span of $\Sigma$ is not contained in $X$ and let $V \subset W$ be the subset of pairs $(x, L)$ such that $L$ is not contained in $X$. Then $t|_U : U \to V$ is invertible. Indeed given any such $(x, L)$, the residual intersection $L \cap X \setminus x$ will be a length two subscheme in $U$. Putting this together, we get the following diagram.

$$\begin{tikzcd}
U \arrow[rr, "\cong"] \arrow[d] & & V \arrow[d] \\
\mathrm{Hilb}^2(X) \arrow[rr, dashed, "t"] & & W \\
Y \arrow[u] \arrow[dr] & & Z \arrow[u] \arrow[dl] \\
& S &
\end{tikzcd} \tag{22}$$

Here $Y$ and $Z$ are the indeterminacy loci of $t$ and $t^{-1}$ respectively. $Y$ is the locus of length two subschemes whose linear span is contained in $X$ and $Y \to S$ sends $\Sigma$ to its linear span. Thus, the fibers of $Y \to S$ are length two subschemes contained on a given line so $Y \to S$ is a $\mathrm{Sym}^2(\mathbb{P}^1) = \mathbb{P}^2$ bundle. On the other hand, $Z$ is the locus of $(x \in L \subset X)$ and $Z \to S$ forgets $x$. Thus $Z \to S$ is the universal family of lines over $S$.

*Remark* 8.32. Diagram 22 is used by Galkin-Shinder [23] to compute many invariants of $S$. For example, one can readily compute from this diagram that $\chi_{top}(S) = 27$ (this is closely related to the 27 lines on a cubic surface) and that the Hodge diamond of $S$ is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 5 & & 5 & \\ 10 & & 25 & & 10. \\ & 5 & & 5 & \\ & & 1 & & \end{array}$$

Most important for us is the fact that $h^{1,0}(S) = h^{1,2}(X) = 5$.

Diagram 22 allows us to directly relate the geometry of $S$ to the geometry of $X$. Clemens and Griffiths also use a degeneration to access $S$. Namely, suppose that $X_t$ is a family of cubic threefolds degenerating to $X_0$ a cubic with a single double point singularity $x_0 \in X_0$.

**Lemma 8.33.** *$X_0$ is rational.*

*Proof.* Projecting from the double point $x_0$ gives a birational map to $\mathbb{P}^3$. □

It turns out that there is a 1-dimensional family of lines through $x_0$ parametrized by a curve $D_0 \subset S_0 = F(X_0, 1)$. Moreover, $S_0$ is a non-normal surface with double locus $D_0$ and whose normalization is isomorphic to $\mathrm{Sym}^2(D_0)$. Using the specialization $S_t \to S_0$ as $t \to 0$, Clemens-Griffiths can analyze the topology of $S_t$ (see [13, Sections 8 & 9]).

### 8.4.2 The Abel-Jacobi map

Next we define the Abel-Jacobi map. Let $X$ be a rationally connected threefold and $Z \to W$ be any algebraic family of curves on $X$ over a smooth projective base $W$ such that $[Z_u] = [Z_v] \in H_2(X, \mathbb{Z})$ for all $u, v \in W$. Note this is always true if for example $Z$ and $W$ are irreducible and $Z \to W$ is surjective.

Recall that the principal polarization on $\mathrm{IJ}(X)$ induces an isomorphism

$$\mathrm{IJ}(X) = \frac{H^{1,2}(X)}{H^3(X,\mathbb{Z})/\mathrm{tors}} \cong \frac{H^{1,2}(X)^\vee}{H_3(X,\mathbb{Z})/\mathrm{tors}}.$$

compatible with Poincaré duality. Fixing a basepoint $u_0 \in W$, for any $u \in W$ we denote by $\sigma_u$ a choice of 3-chain with boundary $[Z_u] - [Z_{u_0}]$. Note that $\sigma_u$ is well defined up to a cycle in $H_3(X, \mathbb{Z})$. Thus we obtain a well defined map

$$AJ_{u_0} : W \to \mathrm{IJ}(X) \qquad u \mapsto \left(\omega \mapsto \int_{\sigma_u} \omega\right).$$

*Remark* 8.34. The point $AJ_{u_0}(u)$ depends only on the 1-cycles $Z_u$ and $Z_{u_0}$ and not on the family $Z \to W$. Another way to phrase this is that there is a group homomorphism

$$\Phi : \mathscr{Z}_1(X)_{hom} \to \mathrm{IJ}(X)$$

from the group of 1-cycles $\Sigma = \sum a_i C_i$ such that the homology class $[\Sigma] = 0$. Then there exists a 3-chain $\sigma$ with $\partial\sigma = \Sigma$ and

$$\Phi(\Sigma) = \left(\omega \mapsto \int_\sigma \omega\right).$$

Thus for any algebraic family of curves $Z \to W$, the Abel-Jacobi map satisfies that

$$AJ_{u_0}(u) = \Phi([Z_u] - [Z_{u_0}]).$$

By universal property of $\mathrm{Alb}(W)$, we obtain a factorization

$$W \to \mathrm{Alb}(W) \to \mathrm{IJ}(X).$$

The corresponding map on lattices is given by

$$H_1(W, \mathbb{Z}) \to H_3(X, \mathbb{Z}) \qquad \gamma \mapsto \sigma_\gamma$$

where $\sigma_\gamma$ is the 3-cycle obtained by considering the total space of the family of curves $\{Z_u\}_{u\in\gamma}$.

**Lemma 8.35.** *Suppose that $Z_u \sim_{rat} Z_{u_0}$, then $AJ_{u_0}(u) = 0$.*

*Proof.* Rational equivalence means that there exists a family of curves $Z' \to \mathbb{P}^1$ such that $Z'_0 = Z_{u_0}$ and $Z'_\infty = Z_u$. On the other hand, $\mathrm{Alb}(\mathbb{P}^1)$ is a point so that induced Abel-Jacobi map $AJ_0 : \mathbb{P}^1 \to \mathrm{IJ}(X)$ must be constant. Thus $0 = AJ_0(\infty) = AJ_{u_0}(u)$. □

Applying this construction to the universal family of lines $Z \to S$ over the Fano surface of lines on $X$, we obtain a map

$$\phi : \mathrm{Alb}(S) \to \mathrm{IJ}(X)$$

On the other hand, $\mathrm{Alb}(S)$ is equipped with a canonical polarization depending only on a choice of basepoint $s_0 \in S$. Let

$$D_s = \{u \in S \mid Z_u \cap Z_s \neq\}.$$

Then we have a map $S \to \mathrm{Pic}^0(S)$ given by

$$s \mapsto \mathscr{O}_S(D_s - D_{s_0}).$$

This induces a factorization $\eta : \mathrm{Alb}(S) \to \mathrm{Pic}^0(S)$. Finally, there is a natural map $\lambda : \mathrm{IJ}(X) \to \mathrm{Pic}^0(S)$ given by dualizing $\phi$ and identifying $\mathrm{IJ}(S)$ with its dual. By analyzing the geometry of $S$ using Diagram 22, one sees that $\phi$ is an isogeny (see Remark 8.32). In fact, much more is true.

**Theorem 8.36.** *[13, Theorems 0.8 - 0.10] Let $X$ be a smooth cubic threefold and $S$ its Fano surface of lines.*

1. *The diagram*

$$\begin{tikzcd} \mathrm{Alb}(S) \arrow[dd, "\eta"'] \arrow[dr, "\phi"] & \\ & \mathrm{IJ}(X) \arrow[dl, "\lambda"] \\ \mathrm{Pic}^0(S) & \end{tikzcd}$$

   *commutes and every map is an isomorphism.*

2. *The map $\psi : S \to \mathrm{Alb}(S)$ is generically injective and its image has cohomology class*

$$[\psi(S)] = \frac{\theta_X^3}{3!}.$$

3. *The induced map $\mathrm{Sym}^2(S) \to \mathrm{Alb}(S)$ is generically finite and its image is isomorphic to $\theta_X$.*

### 8.4.3 Prym varieties and the Torelli theorem

There is another description of the intermediate Jacobian of a cubic threefold $X \subset \mathbb{P}^4$ using Prym varieties.

**Definition 8.37.** Let $\pi : C' \to C$ be an étale double cover between smooth curves. The *Prym variety* $\mathrm{Prym}(\pi)$ is the cokernel of the map

$$\pi^* : \mathrm{Jac}(C) \to \mathrm{Jac}(C').$$

$\mathrm{Prym}(\pi)$ is a principally polarized abelian variety (see e.g. [3]). We denote its theta divisor by $\theta_\pi$.

Let $l \subset X$ be a line on a cubic threefold and let $Y = \mathrm{Bl}_l X$ be the blowup of $X$ along $l$. By Corollary 8.28, $\mathrm{IJ}(X) \cong \mathrm{IJ}(Y)$. The projection $X \dashrightarrow \mathbb{P}^2$ away from $l$ resolves to a conic bundle

$$\mu : Y \to \mathbb{P}^2$$

with discriminant $C \subset \mathbb{P}^2$ of degree 5. Over $C$ we have a family

$$\mu^{-1}(C) \to C$$

of reducible conics. Each fiber over $C$ is a union of two lines and so the family of lines over $C$ gives a 2-to-1 cover

$$\pi : C' \to C.$$

The following Theorem is attributed to Mumford (see [13, Appendix C]).

**Theorem 8.38.** *The Prym variety* $(\mathrm{Prym}(\pi), \theta_\pi)$ *is isomorphic to* $(\mathrm{IJ}(X), \theta_X)$.

*Proof sketch.* Since $X$ and $Y$ have the same intermediate Jacobian, we may replace $X$ with $Y$. By definition, $C'$ is the moduli space of lines in the fibers of $\mu^{-1}(C) \to C$. Thus over $C'$ there is a universal family of lines $\{Z_{c'}\}_{c' \in C'}$ which induces an Abel-Jacobi map

$$C' \to \mathrm{Jac}(C') \to \mathrm{IJ}(X).$$

For $x \in C$, let $l_x, l'_x$ be the two lines lying over $x$. Then for any two points $x, y \in C$,

$$AJ(\pi^* x - \pi^* y) = \Phi(l_x + l'_x - l_y - l'_y) = \Phi(\mu^* x - \mu^* y) = 0$$

since $x$ and $y$ are rationally equivalent on $\mathbb{P}^2$ so their pullbacks are rationally equivalent on $Y$. Therefore, the map

$$\mathrm{Jac}(C') \to \mathrm{IJ}(X)$$

factors through $\mathrm{Jac}(C')/\pi^*\mathrm{Jac}(C) = \mathrm{Prym}(\pi)$. With some work one can show this is an isomorphism of ppavs. □

By analyzing the geometry of this Prym variety $\mathrm{Prym}(\pi)$, Beauville was able to show the following.

**Theorem 8.39.** *[2] The theta divisor $\theta_X$ has a unique singular point $z \in \theta_x$ with projectivized tangent cone isomorphic to $X$.*

As corollaries, one obtains the final main theorems of [13], and in particular the irrationality of a smooth cubic threefold.

**Theorem 8.40.** *[13, Theorems 0.11 & 0.12] Let $X$ be a smooth cubic threefold.*

*(1) The intermediate Jacobian $(\mathrm{IJ}(X), \theta_X)$ is irreducible and not the Jacobian of a curve. In particular, $X$ is not rational.*

*(2) The Torelli Theorem holds for cubic threefolds. That is, $X$ is determined by its intermediate Jacobian $(\mathrm{IJ}(X), \theta_X)$.*

*Proof.* By Theorem 8.39, $\theta_X$ has a unique singular point and in particular must be irreducible. Thus $\mathrm{IJ}(X)$ is an irreducible ppav by Lemma 8.13. Now the theta divisor $\theta_C$ of of a genus 5 curve has singular locus of dimension at least 1 by Theorem 8.23. Thus, $\mathrm{IJ}(X)$ is not a Jacobian. We conclude that $X$ is not rational by Theorem 8.29. The Torelli Theorem follows from the fact that the singularity of the theta divisor determines $X$ (Theorem 8.39). $\square$

# 9 References


[1] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[2] Arnaud Beauville. Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977.

[3] Arnaud Beauville. Prym varieties: a survey. In *Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987)*, volume 49 of *Proc. Sympos. Pure Math.*, pages 607–620. Amer. Math. Soc., Providence, RI, 1989.

[4] Arnaud Beauville, Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Peter Swinnerton-Dyer. Variétés stablement rationnelles non rationnelles. *Ann. of Math. (2)*, 121(2):283–318, 1985.

[5] Roya Beheshti and Eric Riedl. Linear subspaces of hypersurfaces. *Duke Math. J.*, 170(10):2263–2288, 2021.

[6] Olivier Benoist and Olivier Wittenberg. The Clemens-Griffiths method over non-closed fields. *Algebr. Geom.*, 7(6):696–721, 2020.

[7] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013.

[8] F. Campana. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4)*, 25(5):539–545, 1992.

[9] F. Campana, J.-P. Demailly, and Th. Peternell. Rationally connected manifolds and semipositivity of the Ricci curvature. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 71–91. Cambridge Univ. Press, Cambridge, 2015.

[10] Serge Cantat. The Cremona group. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 101–142. Amer. Math. Soc., Providence, RI, 2018.

[11] Andre Chatzistamatiou and Kay Rülling. Higher direct images of the structure sheaf in positive characteristic. *Algebra Number Theory*, 5(6):693–775, 2011.

[12] I. A. Cheltsov. On a smooth four-dimensional quintic. *Mat. Sb.*, 191(9):139–160, 2000.

[13] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[14] Jean-Louis Colliot-Thélène and Alena Pirutka. Hypersurfaces quartiques de dimension 3: non-rationalité stable. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(2):371–397, 2016.

[15] D. F. Coray. Algebraic points on cubic hypersurfaces. *Acta Arith.*, 30(3):267–296, 1976.

[16] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995.

[17] Tommaso de Fernex. Birationally rigid hypersurfaces. *Invent. Math.*, 192(3):533–566, 2013.

[18] Tommaso de Fernex. Erratum to: Birationally rigid hypersurfaces [ MR3049929]. *Invent. Math.*, 203(2):675–680, 2016.

[19] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. Bounds for log canonical thresholds with applications to birational rigidity. *Math. Res. Lett.*, 10(2-3):219–236, 2003.

[20] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[21] Noam Elkies. Complete cubic parametrization of the fermat cubic surface. https://people.math.harvard.edu/ elkies/4cubes.html.

[22] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1984.

[23] Sergey Galkin and Evgeny Shinder. The fano variety of lines and rationality problem for a cubic hypersurface, 2014.

[24] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67, 2003.

[25] Joe Harris, Barry Mazur, and Rahul Pandharipande. Hypersurfaces of low degree. *Duke Math. J.*, 95(1):125–160, 1998.

[26] Jack Huizenga. Rationally connected varieties. `personal.psu.edu/jwh6013/Research/RationalConnectivity.pdf`.

[27] V. A. Iskovskikh and Ju. I. Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. Sb. (N.S.)*, 86(128):140–166, 1971.

[28] Nicholas M. Katz and Peter Sarnak. *Random matrices, Frobenius eigenvalues, and monodromy*, volume 45 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1999.

[29] J Kollár. The rigidity theorem of fano–segre–iskovskikh–manin–pukhlikov–corti–cheltsov–de fernex–ein–mustaţă–zhuang, 2018.

[30] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 1996.

[31] János Kollár. Unirationality of cubic hypersurfaces. *J. Inst. Math. Jussieu*, 1(3):467–476, 2002.

[32] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[33] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[34] Maxim Kontsevich and Yuri Tschinkel. Specialization of birational types. *Invent. Math.*, 217(2):415–432, 2019.

[35] Serge Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952.

[36] J. Lüroth. Beweis eines Satzes über rationale Curven. *Math. Ann.*, 9(2):163–165, 1875.

[37] T. Matsusaka and D. Mumford. Two fundamental theorems on deformations of polarized varieties. *Amer. J. Math.*, 86:668–684, 1964.

[38] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[39] Johannes Nicaise and John Christian Ottem. A refinement of the motivic volume, and specialization of birational types, 2021.

[40] Johannes Nicaise and Evgeny Shinder. The motivic nearby fiber and degeneration of stable rationality. *Invent. Math.*, 217(2):377–413, 2019.

[41] Manuel Ojanguren. *The Witt group and the problem of Lüroth.* Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1990. With an introduction by Inta Bertuccioni.

[42] K Paranjape and V. Srinivas. Unirationality of the general complete intersection of small multidegree. *Astérisque*, 11, 1992.

[43] Thomas Peternell. Kodaira dimension of subvarieties. II. *Internat. J. Math.*, 17(5):619–631, 2006.

[44] Aleksandr V. Pukhlikov. Birational automorphisms of Fano hypersurfaces. *Invent. Math.*, 134(2):401–426, 1998.

[45] Tonny Albert Springer. Sur les formes quadratiques d'indice zéro. *C. R. Acad. Sci. Paris*, 234:1517–1519, 1952.

[46] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.