Math 290: Birational geometry of algebraic varieties

Lecture 1

Class time: 11:00-12:15 EST

Office Hours: TBD

Class Discussion: Discord server

Videos: will be available soon

Grades: Periodically assigned psets due 2 weeks later + final project

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Goal: Classify algebraic varieties up to birational equivalence

$$\begin{tikzcd} X \arrow[r, dashed, shift left, "f"] & Y \arrow[l, dashed, shift left] \\ U \arrow[u, phantom, "\cup"] \arrow[r, "f|_U"] & V \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$f|_U$ is an isomorphism

$/\mathbb{C}$

§1: Background + basic tools

Blowups: $Z \subsetneq X$ closed subvariety

$$\mathrm{Bl}_Z X = \mathrm{Proj}_X \bigoplus_{d \geq 0} I_Z^d \qquad I_Z = \text{ideal sheaf}$$

$$\begin{tikzcd} \mathrm{Exc}(\pi) \arrow[r, phantom, "\rightsquigarrow"] & E \arrow[r] \arrow[d, phantom, "\cap"] & Z \arrow[d, phantom, "\cap"] \\ & \mathrm{Bl}_Z X \arrow[r, "\pi"] & X \end{tikzcd}$$

projective + birational

Linear series: L a line bundle on X

$\emptyset \neq V \subseteq H^0(X, L)$ $\qquad V = \mathrm{Span}(s_0, \dots, s_n)$

$$\varphi_V : X \dashrightarrow \mathbb{P}(V)$$

$$x \longmapsto [V_x] = \{ s \in V \mid s(x) = 0 \}$$

$$x \longmapsto [s_0(x), \dots, s_n(x)]$$

D Cartier divisor

$$\varphi_{|D|} = \varphi_{H^0(X, \mathcal{O}_X(D))}$$

only makes sense if $|D| \neq \emptyset$

$Bs(V) := \{x \in X \mid \varphi_V \text{ is undefined}\}$
$$= \bigcap_{s \in V} V(s)$$

D ample if $\varphi_{|mD|}$ is a closed embedding
$m \gg 0$

D base-point free $\varphi_{|D|}$ is a morphism
(bpf)

D semi-ample if $\varphi_{|mD|}$ is a morphism $m \gg 0$
i.e. if mD is bpf

<u>Chow's Lemma</u>: any variety is birational to a projective variety

<u>Hironaka's theorem</u>: X any variety, then there exists a proper birational $f: X' \to X$ where X' is nonsingular.

<u>New goal</u>:
Classify smooth projective varieties

up to birational equivalence

Canonical divisor

X smooth $\quad \Omega^1_X$ is locally free of rank $\dim X$

$\omega_X = \Lambda^{\dim X} \Omega^1_X$ is a line bundle

K_X is a **canonical divisor** if

$$\mathcal{O}_X(K_X) = \omega_X$$

Ex 1 $\quad \omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

$-K_{\mathbb{P}^n}$ is ample $\quad$ Fano

§2: Overview of the minimal model program

dim 1 (curves)

Fact every curve is birational to a unique smooth projective curve!

Classification scheme (smooth proj)

genus $g = H^0(C, \omega_C)$	K_C	$\mathrm{Aut}(C)$	$\mathrm{Proj}\, R(K_C)$	curvature
0 $\mathbb{P}^1$	antiample	PGL_2	$\emptyset$	>0
1 (elliptic)	$=0$	$C \times$ finite	point	$=0$
≥ 2 (higher genus)	ample	finite	C	<0

<u>Def</u> Canonical ring of a smooth proj X

$$R(K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

$R(K_X) \cong R(K_{X'})$ for $X \dashrightarrow X'$ smooth projective

<u>Question of moduli</u>:

0	pt !
1	1-dim !
$g \geq 2$	$3g-3$ dim moduli

<u>Proposed classification strategy</u>

1) Find a unique (or good) smooth projective representative

2) classify the geometry of this representative using properties of K_X

3) construct moduli spaces that parametrize all representatives within each type in 2)

dim 2 (surfaces)

for dim ≥ 2, there are many smooth projective birational varieties

e.g. $\quad Bl_z X \to X \quad z \in X$ a point

Undo these blowups (blow down)

MMP surfaces

K_{X_m} is nef

$$X = X_0 \to X_1 \to X_2 \dashrightarrow X_m$$

blowups at a point

$X_m \to C$ ruled

$X_m = \mathbb{P}_C(\mathcal{E})$

$(X_m = \mathbb{P}^2)$

X_m not the blowup of a smooth projective surface

(minimal surfaces)

Def 1 if X projective, D cartier

Then D is **nef** if for any

curve $C \subseteq X$, $D.C \geq 0$

$$= \deg(f^* \mathcal{O}_X(D))$$

$$f : C \to X$$

In ruled case:

$X_m \to C$

1) $H^0(dK_X) = 0 \quad d > 0$

2) $\operatorname{Proj} R(K_X) = \emptyset$

3) fibers of $X_m \to C$ are $\mathbb{P}^1$'s (-K - ample) (Fano fibration)

4) X_m is not unique

$$\mathbb{P}_C(\mathcal{E}) \dashleftarrow\dashrightarrow \mathbb{P}_C(\mathcal{E}')$$

for any $\mathcal{E}, \mathcal{E}'$

K_{X_m} nef case:

1) X_m is unique

Thm if K_{X_m} is nef, then it's semiample

$$\varphi_{|dK_{X_m}|} : X_m \longrightarrow Y = \mathrm{Proj}\, R(K_X) = \begin{cases} 2 \text{ dim} \\ 1 \text{ dim} \\ 0 \text{ dim} \end{cases}$$

Def Kodaira dimension of X

$$\kappa(X) = \dim \mathrm{Proj}\, R(K_X) = \max\{\dim \varphi_{|dK_X|}(X) \mid d > 0\}$$

depends on finite generation

$\leq \dim X$

or $= -\infty$

$\kappa = 2$: X_m the minimal model

$$\begin{array}{c} X_m \\ \downarrow \\ X_{can} = Y = \mathrm{Proj}\, R(K_X) \end{array}$$

canonical model

$K_{X_{can}}$ is ample

X_{can} is singular

$\kappa = 1$: $X_m \to Y$ is a genus 1 fibration

elliptic surfaces

K-trivial fibers

$\underline{\kappa = 0}$: $X_m \longrightarrow Y = pt$

dK_{X_m} is trivial for $d \gg 0$

K-trivial case

$\underline{\text{Def}}$ X is
- of general type if $\kappa(X) = \dim X$
- K-trivial if $dK \sim_{\mathbb{Q}} 0$ for some $d > 0$
- Fano if $-K_X$ is ample

These 3 types of varieties form the building blocks for building birational eq classes

$\underline{\text{Thm}}$ (MMP for surfaces)

Any smooth projective surface is birational to one of the following:

a minimal surface of general type $\kappa = 2$

a K-trivial fibration over a curve $\kappa = 1$

X_m unique

K_{X_m} nef

a k-trivial surface $\qquad \kappa = 0$

a fano fibration over a curve $\qquad \kappa = -\infty$ — many minimal surfaces

to construct moduli, need boundedness

Expectation in higher dimensions — for MMP

Hopes/conjecture

$$\begin{array}{ccccccccc} & & & & & & & & K_{X_m} \text{ is nef} \\ & & & & & & & \nearrow & \\ X & \overset{\rho_1}{\dashrightarrow} & X_1 & \overset{\rho_2}{\dashrightarrow} & X_2 & \overset{\rho_3}{\dashrightarrow} & X_3 & \dashrightarrow X_m & \\ & & & & & & & \searrow & \\ & & & & & & & & X_m \xrightarrow{\rho_Z} Z \end{array}$$

Fano fibration (mori fiber space)

$\kappa(X) = -\infty$ $\qquad \dim Z < \dim X$

+ in the K_{X_m} nef case

$$\varphi_{|dK_{X_m}|} : X_m \longrightarrow Z = \operatorname{Proj} R(K_X)$$

φ is a k-trivial fibration

$0 \leq \kappa(X) < \dim X$

φ is birational & K_Z is ample

$\kappa(X) = \dim X$ canonical model

Key features:

1) the maps f are determined by geometry of rational curves on X

Cone + Contraction theorem

2) minimal X_m not unique $(\kappa(X) \geq 0)$

but $Z = \operatorname{Proj} R(K_X)$

3) in the general type case, Z is birational to X, want to construct moduli of finite type indexed by numerical invariants of Z

4) in $\kappa(X) < -\infty$, Fano varieties covered by ratl curves so expect this case to be the wildest case

Issues + difficulties + etc

0) singularities

1) Existence + termination of flip

2) Abundance conjecture: if K_{X_m} is nef then it is semi ample

(good minimal model)

3) Finiteness of minimal models?

4) Classify Fano + K-trivial varieties/ fibrations?

5) Important to generalize to pairs (X, D)