Relative Setting

(Suppose X & Y are normal)

$f: X \to Y$ projective

$N_1(X/Y) = \mathrm{Span}\{[C] \mid f(C) = pt\} \subseteq N_1(X)$

closure

$\overline{NE}(X/Y) = \{\sum \alpha_i [C_i] \mid 0 \le \alpha_i \in \mathbb{R},\ f(C_i) = pt\}$

$N^1(X/Y) = N^1(X) / \equiv_Y$

$D_1 \equiv_Y D_2$ if $D_1 \cdot C = D_2 \cdot C$ for all C s.t. $f(C) = pt$

if Y is $\mathbb{Q}$-factorial + normal, f is birational

$N^1(X/Y) = \mathrm{Coker}(f^*: N^1(Y) \to N^1(X))$

$\mathcal{L}$ line bundle on X

$f^* f_* \mathcal{L} \xrightarrow{\lambda} \mathcal{L} \rightsquigarrow$

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr] & & \mathbb{P}_Y(f_*\mathcal{L}) \arrow[dl] \\ & Y & \end{tikzcd}$$

$x \in X$ if λ_x surjective it defined a 1-dim quotient of $(f_*\mathcal{L})_{f(x)}$

Rmk $f: X \to Y = pt$, $f_*\mathcal{L} = H^0(X, \mathcal{L})$ recovers the usual linear series.

Def 1) $\mathcal{L}$ is f-ample if for some $m \gg 0$, $f^* f_* \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}$ induces an embedding

$$\varphi_{\mathcal{L}^{\otimes m}/Y} : X \hookrightarrow \mathbb{P}_Y(f_* \mathcal{L}^{\otimes m})$$

2) f-base point free, $\varphi_{\mathcal{L}/Y}$ is a morphism

f-semiample, $\varphi_{\mathcal{L}^{\otimes m}/Y}$ morphism for $m \gg 0$

3) D is f-effective if

$$f_* \mathcal{O}_X(D) \neq 0$$

4) f-nef: $D.C \geq 0$ for all $[C] \in \overline{NE}(X/Y)$

5) D is f-big if $D|_{X_\eta}$ is big where X_η is the generic fiber

$\operatorname{rk} f_* \mathcal{O}_X(mD)$ has maximal growth rate.

Fact: relative Kodaira's Lemma:

D f-big $\iff$ $D = A + E$

$A = f$-ample

$E = f$-effective

Prop: Suppose $D|_{X_y}$ is ample

[KM 1.41] then D is f-ample over a neighborhood of $y \in U \subseteq Y$

Thm 1) relative Nakai - Moishezon:

D is f-ample $\iff$ for all varieties W s.t. $f(W) = pt$

$$D^{\dim W} \cdot W > 0$$

2) relative Kleiman Theorem:

D f-nef $\Rightarrow$ $D^{\dim W} \cdot W \geq 0$

for all W s.t. $f(W) = pt$

3) relative Kleiman criterion:

D f-ample $\iff$

$$N_1(X/Y) \supseteq D_{>0} \supseteq \overline{NE}(X/Y)$$

4) if A is f-ample, H is ample on Y then $A + mH$ ample for $m \gg 0$

Divisors on Singular Varieties

X normal quasi projective (until further notice)

$$WDiv(X)_K = \{ \sum \alpha_i D_i \mid D_i \text{ pure codim } 1,\ \alpha_i \in K \}$$

$K = \mathbb{R}, \mathbb{Q}$

$D_1 \sim D_2$ iff $D_1 = D_2 + \operatorname{div}(f)$, f rational

$$|D| = \{ f \mid \operatorname{div}(f) + D \geq 0 \} / \text{scaling}$$

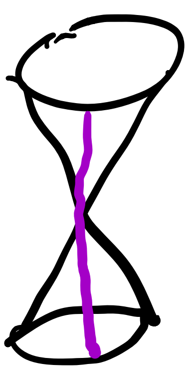
$D_1 \sim_K D_2$ if $rD_1 \sim rD_2$ $r \in K$

$$Bs|D| = \bigcap_{D' \in |D|} D',$$

K-cartier divisors

$\cap$

$$WDiv(X)_K / \sim_K \supseteq Div(X)_K / \sim_K$$



D

$y^2 = xz$

$2D = V(x)$

$D = V(y, x)$

$\implies$ D is $\mathbb{Q}$-Cartier but not Cartier Weil div

Weil divisorial sheaves:

$$\mathcal{O}_X(D)(U) = \{ f \mid (\mathrm{div}(f) + D)|_U \geq 0 \}$$

$$|D| = \mathbb{P} H^0(X, \mathcal{O}_X(D))$$

$\mathcal{O}_X(D)$ is not a line bundle if D is not Cartier, but its a rank 1 reflexive sheaf

<u>Def</u> $\mathcal{F}$ is reflexive if

$\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism

where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$

<u>Key Property</u> reflexive sheaves on normal varieties are S_2 (Serre's condition)

i.e. $\mathcal{F}(U) = \mathcal{F}(U \setminus Z)$ $\quad \mathrm{codim}(Z \subseteq U) \geq 2$

$\Rightarrow$ if $U \overset{j}{\subseteq} X$ big open (complement has codim ≥ 2)

then $\mathcal{F} = j_* \mathcal{F}|_U$

$$\mathcal{O}_X(mD) = \left(\mathcal{O}_X(D)^{\otimes m}\right)^{\vee\vee} =: \mathcal{O}_X(D)^{[m]}$$

<u>$\mathbb{Q}$-line bundle</u>: rank 1 reflexive sheaf $\mathcal{F}$ s.t. $\mathcal{F}^{[m]}$ is a line bundle for some $m > 0$

$$\begin{array}{ccc} & D \mapsto \mathcal{O}_X(D) & \\ \left\{\begin{array}{c}\text{lin or eq} \\ \text{classes of} \\ \text{Weil divisors}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{rank } 1 \\ \text{reflexive} \\ \text{sheaves}\end{array}\right\} \\ \cup\!\! \mid & & \cup\!\! \mid \\ \left\{\begin{array}{c}\mathbb{Q}\text{-Cartier} \\ \text{Divisors up} \\ \text{to } \sim\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\mathbb{Q}\text{-line} \\ \text{bundles}\end{array}\right\} \end{array}$$

$\mathbb{Q}$-factorial: every Weil Divisor is $\mathbb{Q}$-cartier

<u>Def</u> $K_X = j_* K_U$ $\quad U \subseteq X$ big open smooth subvariety

$\omega_X = j_* \omega_U = \mathcal{O}_X(K_X)$

$\mathbb{Q}$ - Gorenstein: K_X is $\mathbb{Q}$-Cartier

Warning: $\mathbb{Q}$-Gorenstein does not imply Cohen-Macaulay

$$\omega_X[\dim X] = \mathcal{H}^{-\dim X}(\omega_X^\bullet)$$

but there can be other cohomology sheaves

Log resolutions

$\mu : X \to Y$ proper birational

$\mu_*^{-1}(D)$ = strict transform of $D \in WDiv(Y)$

A log resolution of (Y, D) is a proper birational $\mu : X \to Y$ s.t. 1) X smooth

2) $\mu_*^{-1}(D) \cup Ex(\mu)_{red}$ has simple normal crossings

i.e. each components are smooth

and they intersect as $x_1 \cdots x_m = 0$

Singularities of the MMP

Canonical singularities:

Y is a normal + $\mathbb{Q}$-Gorenstein s.t. for any resolution

$f: X \to Y$, $f_* \mathcal{O}_X(mK_X) = \mathcal{O}_Y(mK_Y)$

where mK_Y is Cartier

Rmk only need to check on a single resolution

Fact if K_Y ample then $Y = \text{Proj } R(K_X)$, i.e. its the canonical model of X

Prop Suppose Y normal & mK_Y is Cartier, then

Y has canonical singularities

$$\iff K_X \sim_{\mathbb{Q}} f^* K_Y + \sum a_i E_i$$

where $f: X \to Y$ is a resolution $a_i \geq 0$ effective exceptional

<u>Proof</u> $\Longrightarrow$:

$$\mathcal{O}_Y(mK_Y) \xrightarrow{\sim} f_*\mathcal{O}_X(mK_X)$$

$$f^*\mathcal{O}_Y(mK_Y) \hookrightarrow \mathcal{O}_X(mK_X) \qquad (*)$$

iso at generic point $\Longrightarrow$ injective

$$\Longrightarrow \quad \mathcal{O}_X(mK_X) = f^*\mathcal{O}_Y(mK_Y) \otimes \mathcal{O}_X(E)$$

effective and exceptional

$$mK_X \sim f^*mK_Y + E$$

$$K_X \sim_{\mathbb{Q}} f^*K_Y + \sum a_i E_i \qquad (**)$$

E_i exceptional & $a_i \geq$

$\Longleftarrow$: Suppose $(**)$

$$0 \to f^*\mathcal{O}_Y(mK_Y) \to \mathcal{O}_X(mK_X) \to \mathcal{O}_E(E) \to 0$$

$$f_m^*K_Y|_E = 0 \quad \text{so} \quad mK_X|_E = E|_E$$

by $(**)$

push forward

$$0 \to \cancel{f_*} f^* \mathcal{O}_Y(mK_Y) \to f_* \mathcal{O}_X(mK_X) \to f_* \mathcal{O}_E(E)$$

by ZMT, $f_* \mathcal{O}_X \cong \mathcal{O}_Y$

so $f_* f^* \mathcal{L} = f_*(f^* \mathcal{L} \otimes \mathcal{O}_X)$

$= \mathcal{L} \otimes f_* \mathcal{O}_X = \mathcal{L}$

<u>Claim</u> $f_* \mathcal{O}_E(E) = 0$

$\dim X = 2$: $E^2 < 0$ so $\mathcal{O}_E(E)$ has negative degree

$\dim X > 0$, slice by hyperplanes and induct $\boxtimes$

Suppose $f: X \to Y$ is a divisorial extremal contraction, $\mathrm{Ex}(f) = E$, suppose X is smooth, mK_Y Cartier

$$K_X \sim_{\mathbb{Q}} f^* K_Y + aE$$

$0 > K_X . C = aE . C \Rightarrow a > 0$
for curve $C \subseteq$ fiber $\subseteq E$ $\quad E . C < 0$

Def Y has terminal if its normal, $\mathbb{Q}$-Gorenstein & for any resolution $f: X \to Y$

$$Ex(f) = \bigcup_{i=1}^{n} E_i,$$

$$K_X \sim_{\mathbb{Q}} f^* K_Y + \sum a_i E_i$$

where $a_i > 0$