Def Y has terminal singularities if K_Y is $\mathbb{Q}$-Cartier and for any resolution $f: X \to Y$

$$K_X \sim_{\mathbb{Q}} f^* K_Y + \sum a_i E_i \qquad a_i > 0$$

$$E = \mathrm{Exc}(f) = \bigcup E_i \qquad a_i = a(E_i, Y)$$

discrepancies

Example if Y is smooth, then Y has terminal singularities

$$\begin{array}{ccccc} & & X'' & & \\ & \swarrow & & \searrow & \\ X & & & & X' \\ & \searrow & & \swarrow & \\ & & Y & & \end{array} \implies \text{independent of choice of resolution}$$

$k(Y)$ = function field of Y

$E \subseteq X \xrightarrow{f} Y$ $\quad f$ birational, $E \subseteq X$ divisor

$\mathrm{ord}_E f = \nu(E, X)$: $k(X) = k(Y)$ is a discrete valuation

$\nu(E, X) \Rightarrow a(E, Y)$

but if $\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ {\scriptstyle g}\downarrow & \nearrow{\scriptstyle f} & \\ X & & \end{array}$ $\quad g$ isomorphism over the generic point of E, E'

$$\nu(E', X') = \nu(E, X)$$

$$\Rightarrow a(E, Y) = a(E', Y)$$

E is a divisor lying over Y up to equivalence coming from inducing the same valuation $N(E, x)$

$$\text{center}_Y(E) = \text{closure of } f(E) \subseteq Y$$

<u>Lemma</u> Y has terminal singularities if and only if $\exists$ some $f: X \to Y$ resolution s.t.

$$f_* \mathcal{O}_X(mK_X - E) \cong \mathcal{O}_Y(mK_Y)$$

mK_Y is Cartier & $E = \sum E_i$ reduced exceptional

<u>Proof</u> $mK_X = f^* mK_Y + \sum ma_i E_i$

$\mathbb{Z} \ni ma_i > 0$ $\quad ma_i \geq 1$

$$mK_X - E = f^* mK_Y + \underbrace{\sum (ma_i - 1) E_i}_{\text{effective}}$$

$ma_i - 1 \geq 0$, then proceed as before

<u>Summary</u> determined by $a(E, Y)$

smooth $\subseteq$ terminal $\subseteq$ canonical

Ex 1) terminal surfaces are smooth.

$f: X \to Y$ resolution, but Y is terminal, $\dim Y = 2$

$E = \bigcup E_i$ $(E_i . E_j)$ negative definite

$$K_X \sim_{\mathbb{Q}} f^* K_Y + \sum a_i E_i \qquad \underline{a_i > 0}$$

negative-definite $\Longrightarrow$ there exists E_j

s.t. $\left(\sum_i a_i E_i\right) . E_j < 0$

$$K_X . E_j = \left(f^* \cancel{K_Y}^{\,0} + \underbrace{\sum a_i E_i\right) . E_j}_{<0}$$

< 0

$E_j^2 < 0 \quad \Longrightarrow \quad E_j \cong \mathbb{P}^1$ is a (-1)-curve

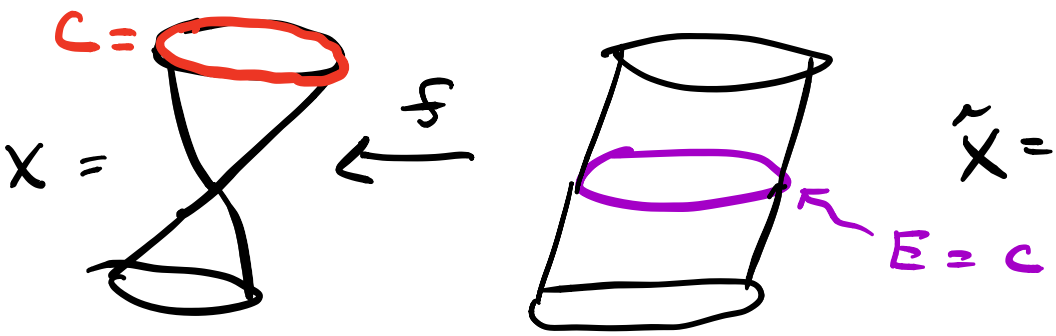
So our resolution factors as

$$X \xrightarrow{g} X' \xrightarrow{f'} Y$$

blowup of a point & $E_j = \mathrm{Exc}(g)$, X' is smooth by Castelnuovo

2) $C \subseteq \mathbb{P}^2$ degree n curve

$$X = \mathrm{Cone}(C \subseteq \mathbb{P}^2) \subseteq \mathbb{A}^3$$



$$\tilde{X} = \mathrm{Tot}_C(\mathcal{O}(-1))$$

$$E^2 = \deg N_{E/\tilde{X}} = \deg_C \mathcal{O}(-1) = -n$$

$$(K_{\tilde{X}} + E)\big|_E = K_E = 2g-2 = n(n-3)$$

$$K_{\tilde{X}} = f^* K_X + aE$$

$$K_{\tilde{X}} . E = n(n-3) + n = n(n-2)$$

$$\phantom{K_{\tilde{X}} . E = n(n-3) + n =}\ \ \ \ \ \ \ \| $$

$$(f^* K_X + aE) . E = -an$$

$$a = 2-n$$

<u>$n=1$</u> smooth cone $a=1$

<u>$n=2$</u> quadric cone $y^2 - xz$ $a=0$

canonical singularity

<u>$n=3$</u> log canonical sing $a=-1$

<u>$n \geq 4$</u> bad

3) $S \subseteq \mathbb{P}^3$ of degree n

$\mathrm{Tot}\,\mathcal{O}(-1) = \tilde{X} \to X =$ cone

$\cup$

$E \cong S$

log discrepancy

$$K_{\tilde{X}} + E = f^* K_X + \underbrace{(1+a)}E$$

$$(K_{\tilde{X}} + E)|_E = K_S = \mathcal{O}_S(n-4) = (K_{\mathbb{P}^3} + S)|_S$$

$$(f^* K_X + (1+a)E)|_S = (1+a)\, E|_E$$

$$= \mathcal{O}_S(-1-a)$$

$1 + a = 4 - n \qquad a = 3 - n$

<u>$n=2$</u> Cone over quadric surface

$\{xy - zw = 0\} \subseteq \mathbb{A}^4$

but $a = 3 - 2 = 1$

singular but terminal

<u>$n=3$</u> $a = 0$ cone (cubic)

canonical singularities

klt sing

<u>$n=4$</u> log canonical sing $a=-1$

<u>$n \geq 5$</u> bad (klt singularities are rational)

log pairs

(X, D) $\qquad D = \sum a_i D_i \quad a_i \in \mathbb{Q}$

normal
irr, q-proj

prime

$\mathbb{Q}$-Weil divisor

Def 1) (X, D) is a log pair if $K_X + D$ is $\mathbb{Q}$-Cartier

2) D is a boundary if $0 \le a_i \le 1$

round up $\lceil D \rceil = \sum \lceil a_i \rceil D_i$ $\in$ Weil divisors

round down $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$

D is reduced if $D = \lfloor D \rfloor$ & D is boundary

$K_X + D$ will be the main focus generalizing K_X if $D = 0$

log canonical divisor

if X is smooth
$\quad \cup\!\mid$
$\quad D$ is smooth

$\Omega^1(\log D) \ni \frac{dx_1}{x_1} \wedge \dots \wedge dx_n$

$\mathcal{O}_X(K_X + D) = \omega_X(D) = \bigwedge^{\dim X} \Omega^1_X(\log D)$

sheaf of $\dim X$-forms with log poles along D

$D = \{x_1 = 0\}$

<u>Philosophy</u> U smooth but not proper

$U \subseteq X$ smooth compactification with $D = X \setminus U$ simple normal crossings, then $\Omega^1_X(\log D)$ are invariants of U

Motivation for us

1) flexibility: a) e.g. $K_X + D$ is $\mathbb{Q}$-Cartier even if K_X not,

b) if $K_X \equiv 0$ but we can add some boundary $D \subseteq X$ and now log mmp for $K_X + D$ can help understand X

c) adjunction $(K_X + D)|_D = K_D$

induction on dimension

lets us relate log mmp of pairs (X, D) in dim $n+1$ to mmp in dim D

d) canonical bundle formula for K-trivial fibrations

(log) discrepencies

(X, D) log pair

$f : Y \to X$ log resolution

$(f_*^{-1} D \cup \mathrm{Exc}(f))$ simple normal crossings

$E = \bigcup E_i$

$Y \setminus E \cong X \setminus f(E)$

$$f^* \mathcal{O}_X(m(K_X + D))\big|_{Y \setminus E} \cong \mathcal{O}_Y(m(K_Y + f_*^{-1} D))\big|_{Y \setminus E}$$

$\exists$ unique $a(E_i, X, D) \in \mathbb{Q}$

s.t.

$$m(K_Y + f_*^{-1} D) \sim f^*(m(K_X + D)) + \sum_{E_i \text{ exc}} m\, a(E_i, X, D) E_i$$

$$K_Y + f_*^{-1} D \sim_{\mathbb{Q}} f^*(K_X + D) + \sum_{E_i \text{ exc}} a(E_i, X, D) E_i$$

$a(E_i, X, D)$ is the discrepency of the exceptional E_i

$P \subseteq X$ prime divisor

not exceptional $a(P, X, D) = -a_i$ if $P = D_i$

or 0 else

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + D) + \underbrace{\sum_{p:\text{prime}} a(p, X, D)\, p}_{A_Y(X,D)}$$

$$b(E_i, X, D) = 1 + a(E_i, X, D)$$

log discrepancy

$(Y, f_*^{-1}D + E)$

$$K_Y + f_*^{-1}D + E \sim_{\mathbb{Q}} f^*(K_X + D) + \sum_{E_i:\text{exc}} b(E_i, X, D) E_i$$

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using numerical equivalence

$$K_Y \equiv f^*(K_X + D) + A$$

$A_Y(X, D)$ is uniquely by being the unique A s.t.

$$f_* A = -D \quad \text{(negativity lemma)}$$

Exercises:

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1) D' effective $\mathbb{Q}$-Cartier divisor, for any prime p lying over X, $a(p, X, D) \geq a(p, X, D + D')$

strict iff $\text{center}_X(p) \subseteq D'$

2) X smooth, $Z \subseteq X$ irreducible

$$\begin{tikzcd} Bl_Z X \arrow[r, "p"] & X \\ E \arrow[u, phantom, "\cup"] \arrow[r] & Z \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$D = \sum a_i D_i$

$a(E, X, D) =$

$k - 1 - \sum a_i \operatorname{mult}_Z D_i$

$k = \operatorname{codim}(Z \subseteq X)$

3) $f: Y \to X$ proper birational

D_Y, D_X s.t.

$f_* D_Y = D_X$ & $K_Y + D_Y = f^*(K_X + D_X)$

then for any prime P lying over X,

$$a(P, Y, D_Y) = a(P, X, D_X)$$