(X, Δ) log pair $\Delta = \sum a_i D_i$

$K_X + \Delta$ is $\mathbb{Q}$-Cartier

$f: Y \to X$ log resolution

discrep of E

$$K_Y = f^*(K_X + \Delta) + \sum a(E, X, \Delta) E$$

$$\mathrm{discrep}(X, \Delta) = \inf_{E \text{ exceptional}} \{a(E, X, \Delta)\}$$

$$\mathrm{total\,discrep}(X, \Delta) = \inf_{P \text{ prime divisors}} \{a(P, X, \Delta)\}$$

Ex 1) Computation of the discrep of exceptional divisor of

$$Bl_Z X \to X$$

2) $f: Y \to X$ proper birational

Δ_Y, Δ_X

s.t. $f_* \Delta_Y = \Delta_X$

$$K_Y + \Delta_Y = f^*(K_X + \Delta_X)$$

then for any P, lying over X $\quad a(P, Y, \Delta_Y) = a(P, X, \Delta_X)$

rmk when $P \subseteq Ex(f)$

$$a(P, Y, \Delta_Y) = -\mathrm{Coeff}(P, \Delta_Y)$$
$$a(P, X, \Delta_X) =$$

Observations: $\Delta = \sum a_i D_i$ (X, Δ) log pair

1) $\mathrm{discrep}(X, \Delta) \leq 1$

pick some smooth codim 2 point of X away from Δ

Bl_{pt} surface compute $a(\text{exceptional}) = 1$

2) If total discrep < -1

then there exists P lying over X with discrep $a(P, X, \Delta) < -k$ for any k

total discrep $< -1 \Rightarrow$ total discrep $= -\infty$

Pick some E lying over X w/ $a(E, X, \Delta) = -1-c$ $c > 0$

let $f: Y \to X$ proper birational

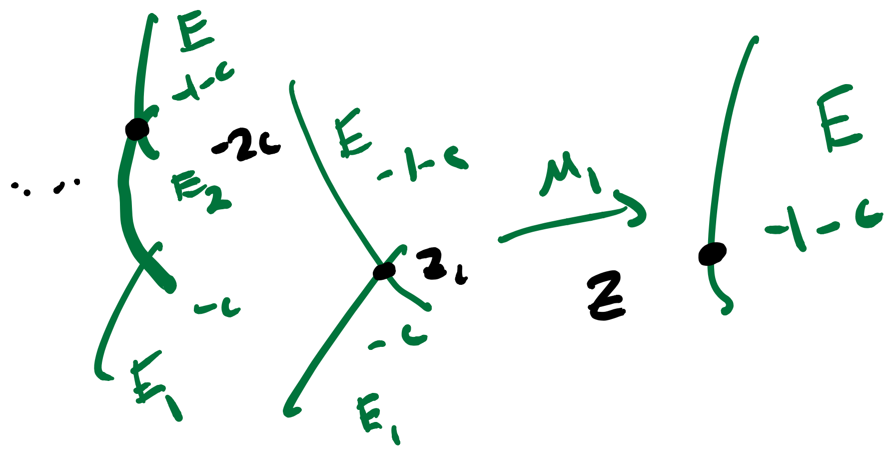
s.t. E is a divisor on Y

e.g. $\mathrm{center}_Y(E)$ is a divisor

$$f^*(K_X+\Delta) = K_Y + \Delta_Y \quad \text{w/} \quad f_*\Delta_Y = \Delta$$

$$a(P, X, \Delta) = a(P, Y, \Delta_Y)$$

$\Delta_Y = (1+c)E$ + exceptional + strict transform of Δ



$Y_1 = Bl_Z Y$

compute on a smooth surface

$K_X - E_1 + (1+c)E_1 \quad q_1^*(K_Y + (1+c)E))$

$+ (1+c)E$

$\Delta_{Y_1} = cE_1 + (1+c)E \qquad a(E_1, Y_1, \Delta_{Y_1}) = -c$

$Y_2 = Bl_{Z_1} Y_1$

Same computation

$\Rightarrow a(E_2, Y_1, \Delta_{Y_1})$

$= -2c$

$\Rightarrow$ Constructed a sequence of

blowups w/ $a(E_k, X, \Delta) = -kc$
$c>0$

3) if (X, Δ) is snc Δ is a boundary
then the discrep

$$\mathrm{disc}(X,\Delta) = \min\{\min_{D_i \cap D_j = \emptyset}\{1-a_i-a_j\}, \min_j\{1-a_j\}, 1\}$$

$= r(X, D)$

if $\Delta = 0$ $\mathrm{discrep}(X) = 1$

if Δ reduced & smooth $\mathrm{discrep}(X,\Delta) = 0$

if Δ reduced but not smooth $\mathrm{discrep}(X,\Delta) = -1$

Sketch: $\mathrm{discrep}(X, D) \leq r(X, D)$

For the other inequality, use the formula for discrepency of blowups +

Lemma: If E over X is any divisor, then there exists a sequence of blowups s.t. $E \subseteq Y \to X$

$Y = X_m \to X_{m-1} \to \dots \to X_1 \to X_0$

+ induction on m to get that $a(E, X, \Delta) \geq r(X, D)$

Prop There exists a log resolution $f: Y \to X$ with exceptionals E_i s.t. $\sum f_*^{-1} D_i$ is smooth. If

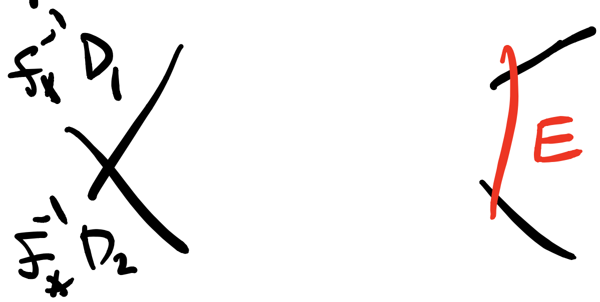
(X, Δ)

$\Delta = \sum a_i D_i$

$a(E_i, X, \Delta) \geq -1$, then

$$\mathrm{discrep}(X, \Delta) = \min\{\min\{a(E_i, X, \Delta)\}, \min_j\{1 - a_j\}, 1\}$$

Pf given a log resolution blowup the intersections of $f_*^{-1} D_i$



Pick Δ_Y s.t. $f_* \Delta_Y = \Delta$

$$f^*(K_X + \Delta) = K_Y + \Delta_Y$$

$b_i = -a(E_i, X, \Delta) =$ coeff of E_i in Δ_Y

$$\mathrm{discrep}(X, \Delta) = \min\{\mathrm{discrep}(Y, \Delta_Y), \min\{-b_i\}\}$$

$$\operatorname{discrep}(Y, \Delta_Y) = \min \{1 - b_i,\ 1 - a_j,\ 1 - b_i - a_j,\ 1 - b_i - b_k,\ 1 - \cancel{a_j} - a_\ell\}$$

$b_i \leq 1 \Rightarrow 1 - b_i \geq 0$

$-b_i \leq$

$$\Rightarrow \operatorname{discrep}(X, \Delta) = \min\{\{1 - a_j\}, a(E_i, X, \Delta)\}$$

Rmk if $\Delta = 0$, for any fixed log resolution $f: Y \to X$

if $a(E_i, X) \geq -1$

$$\Rightarrow \operatorname{discrep}(X) = \min\{a(E_i, X)\}$$

Cor (lower semicontinuity)

(X, Δ) log pair, L Cartier divisor

$L_0 \in |L|$, $L_g \in |L|$ generic

then $\operatorname{discrep}(X, \Delta + cL_0) \leq \operatorname{discre}(X, \Delta + cL_g)$

PF $f: Y \to X$ log resolution

1) $f^*|L| = B + |F|$ ← base point free

2) $f_*^{-1}\Delta$ is smooth

3) $B + f_*^{-1}\Delta + Ex(f)$ is snc

compute discrep on this resolution

$$f^*(K_X + \Delta + cL_\lambda) = K_Y + \Delta_Y + cF_\lambda$$

$$\lambda = 0, g$$

bertini theorem $\Rightarrow$ $\Delta_Y + F_g$ is snc

previous lemma $\Rightarrow$

$$\operatorname{discrep}(Y, \Delta_Y + F_g)$$

$$= -C(g)$$

$C(\lambda)$ = maximal coefficient of an exceptional of f in $\Delta_Y + F_\lambda$

use semicontinuity of multiplicity

$\Rightarrow \quad C(0) \geq C(g)$

$$\operatorname{discrep}(X, \Delta + ch_0) \leq -C(0) \leq \operatorname{discrep}(X, \Delta + ch_g)$$

Def (Singularities of pairs)

Let (X, Δ) be a log pair

$\Delta = \sum a_i D_i$, (X, Δ) is

terminal
$\cap$
canonical
$\cap$
Kawamata log terminal (klt)
$\cap$
Purely log terminal (plt)
$\cap$
log canonical (lc)

if $\operatorname{discrep}(X, \Delta)$
> 0
≥ 0
> -1 + $\lfloor \Delta \rfloor \leq 0$
> -1
≥ -1

klt
plt
lc

log discrep(X, Δ)
$> 0 + \lfloor \Delta \rfloor = 0$
> 0
≥ 0

$$K_Y + \sum E_i + f_*^{-1}(\Delta) = f^*(K_X + \Delta) + B(X, \Delta)$$

log canonical = Proj log canonical ring $R(K_Y + \sum E_i + f_*^{-1}\Delta)$

plt $\leq$ divisorially log terminal $\leq$ lc

klt closed under taking $(K_X+\Delta)$- extremal i.e. the log minimal model program + well adapted to vanishing theorems

plt good for induction:

$\lfloor \Delta \rfloor = S$ ← irreducible divisor

then adjunction $\Rightarrow$

$$(K_X+\Delta)|_S = K_S + \Theta \qquad \Theta \in WDiv_{\mathbb{Q}}(S)$$

$\Delta = S + \underset{\text{coeff} < 1}{\text{stuff}}$

(X,Δ) plt $\Longrightarrow$ (S,Θ) is klt

Def (X,Δ) log pair w/ boundary $\Delta = \sum a_i P_i$ ($0 \leq a_i \leq 1$) then we say (X,Δ) is dlt if there exists $Z \subseteq X$ s.t.

1) $(X \setminus Z, \Delta \setminus Z)$ is snc

2) for any E lying X with

$\mathrm{center}_X(E) \subseteq Z$, $a(E, X, \Delta) > -1$