Singularities of the mmp

(X,Δ) log pair — $\text{discrep}(X,\Delta) = a(X,\Delta)$

$\Delta = 0$

- terminal: $a > 0$
- $\Downarrow$
- canonical: $a \geq 0$
- $\Downarrow$
- klt: $a > -1$, $\lfloor \Delta \rfloor = 0$
- $\Updownarrow$
- plt: $a > -1$
- $\Updownarrow$
- dlt: $\sum a_i D_i = \Delta$, $0 \leq a_i \leq 1$; $\exists\, Z \subseteq X$ s.t. $(X \setminus Z, \Delta \setminus Z)$ is snc & $a(E,X,\Delta) > -1$ for any E s.t. $\text{center}_X(E) \subseteq Z$
- $\Downarrow$
- lc: $a \geq -1$

lt: klt, plt, dlt

general case

- terminal $\Rightarrow$ canonical
- terminal $\Rightarrow$ klt
- canonical $\Rightarrow$ plt
- klt $\Rightarrow$ plt
- plt $\Rightarrow$ dlt
- dlt $\Rightarrow$ lc

Continuity properties

(X,Δ) log pair, Δ effective, D effective $\mathbb{Q}$-Cartier

1) $(X, \Delta + D)$ is term (resp can, klt, plt, dlt, lc) then (X,Δ) is term (resp can, klt, plt, dlt, lc)

2) (X,Δ) is term (resp klt) then $(X, \Delta + \varepsilon D)$ is term (resp klt) for $0 < \varepsilon \ll 1$

3) (X, Δ) is plt & D doesn't share components with Δ then $(X, \Delta + \varepsilon D)$ is plt for $0 \leq \varepsilon \ll 1$

4) if (X, Δ) is dlt & $\mathrm{Supp}(D) \subseteq \mathrm{Supp}(\Delta - \lfloor \Delta \rfloor)$ then $(X, \Delta + \varepsilon D)$ is dlt for $0 \leq \varepsilon \ll 1$

5) if (X, Δ) is terminal (resp klt, plt, dlt) then $(X, \Delta + D)$ is canonical (log canonical) $\iff$ $(X, \Delta + cD)$ is terminal (klt, plt, dlt) for all $c < 1$

<u>Prop</u> (properties of dlt)

1) (X, Δ) is dlt $\iff$ $\exists$ $Z \subseteq X$ s.t. $(X \setminus Z, \Delta \setminus Z)$ is snc & $\exists$ an $f : Y \to X$ s.t. $a(P, X, \Delta) > -1$ for $P \subseteq f^{-1}(Z)$

2) (X, Δ) is dlt $\iff$ there exists a log resolution $f : Y \to X$ s.t. $a(E, X, \Delta) > -1$ for all f-exceptional E

3) if (X, Δ) is dlt then its klt
$\iff$ $\lfloor \Delta \rfloor = 0$

<u>Def</u> let (X, Δ) be a log pair

. non klt place of (X,Δ) is
a divisor P lying over X with
$a(P, X, \Delta) \leq -1$

• non klt center is a subvariety of
the form $\text{center}_X(P)$ for
P a non klt place

• if (X,Δ) is lc, then we call
a non klt center an lc center

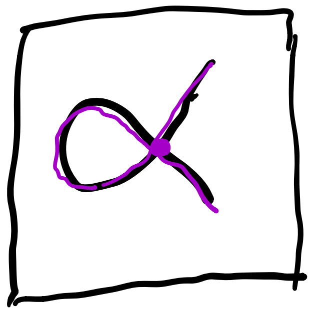
• $\text{Nklt}(X,\Delta) = \bigcup$ non klt centers $\subseteq X$

<u>Def</u> (log canonical threshold)
if (X, Δ) log pair & D effective $\mathbb{Q}$-Cartier
suppose (X, Δ) is lc
$\text{lct}(X,\Delta; D) = \sup\{ t>0 \mid (X, \Delta + tD) \text{ is lc}\}$

Rmks 1) all these definitions make sense for $\mathbb{R}$-divisors ($\mathbb{R}$-Cartier) and a priori this $t \in \mathbb{R}$

2) if Δ, D are both $\mathbb{Q}$-divisors then $t \in \mathbb{Q}$

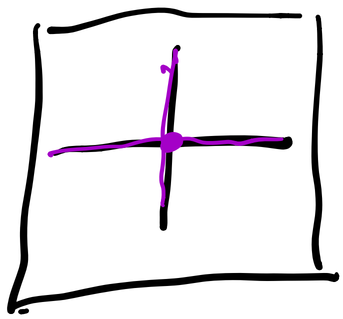
Ex 1) $(\mathbb{P}^2, D) =$



log canonical but not dlt

nklt

2) $(\mathbb{P}^2, D_1 + D_2)$



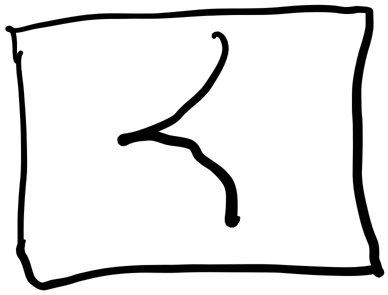
dlt but not plt

3) $(\mathbb{P}^2, D_1 + (1-\varepsilon) D_2)$ plt but not klt

4) $(\mathbb{P}^2, aD_1 + bD_2)$ klt

$0 \le a, b < 1$

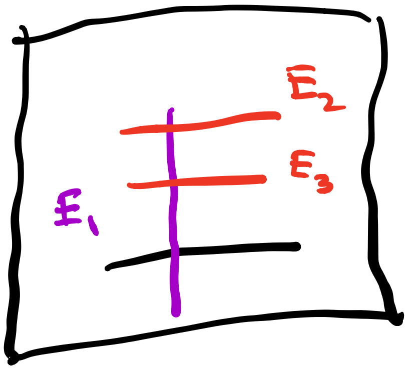
5) $(\mathbb{P}^2, \Delta)$



$y^2 - x^3 = 0$

$\Delta = cD$

$f \uparrow$ blowup 3 times



$E_i^2 = -i$

$f^*D = f_*^{-1}D + 2E_3 + 3E_2 + 6E_1$

$f^*K_X = K_Y - E_3 - 2E_2 - 4E_1$

$$K_Y + \sum E_i + f_*^{-1}(cD)$$

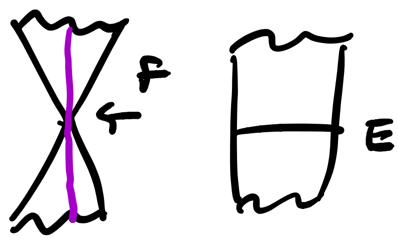
$$= f^*(K_X + cD) + (2-2c)E_3 + (3-3c)E_2 + (5-6c)E_1$$

log disc ≥ 0

$\mathrm{lct}(X, 0; D) = 5/6$

(X, cD) is klt for $c < 5/6$

6)

$X =$ 

cone over a degree n rat'l normal curve

D

(X, Δ) $\qquad E^2 = -n$

log disc

$\Delta = c D$

$$K_Y + E + f_*^{-1}\Delta = f^*(K_X + \Delta) + \frac{2-c}{n} E$$

(X, Δ) is klt for all $c < 1$

its plt for $c = 1$

$$\operatorname{lct}(X, 0; \Delta) = 1$$

Kodaira Vanishing

$/\mathbb{C}$

dim $X = n$

Thm let X be smooth projective L ample Cartier divisor, then

adjoint divisor

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \text{for } i > 0$$

$\Updownarrow$ SD

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for } i < n$$

(false in positive char in general)

Proof

Step 1 $L = H$ is a smooth effective divisor

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

$$H^{i-1}(\mathcal{O}_X) \xrightarrow{res_{i-1}} H^{i-1}(\mathcal{O}_H) \to H^i(\mathcal{O}_X(-H)) \to H^i(\mathcal{O}_X) \xrightarrow{res_i} H^i(\mathcal{O}_H)$$

Want res_i to be isomorphisms for $i < n-1$, injective for $i = n-1$

Lefschetz hyperplane theorem

if $H \subseteq X$ smooth ample divisor, X smooth projective

$$H^i(X, \mathbb{C}) \xrightarrow{res_i} H^i(H, \mathbb{C})$$ isom $i < n-1$, inj for $i = n-1$

$$\cong \qquad\qquad \cong$$

Hodge theory $$\bigoplus_{p+q=i} H^q(\Omega^p_X) \xrightarrow{\oplus res_{p,q}} \bigoplus_{p+q=i} H^q(\Omega^p_H)$$

$$\Longrightarrow \quad r_{p,q} : H^q(\Omega^p_X) \to H^q(\Omega^p_H)$$

isom for $p+q < n-1$

inj for $p+q = n-1$

so when $p=0$, we get

$$H^i(X, \mathcal{O}_X) \to H^i(H, \mathcal{O}_H)$$

isom for $i < n-1$

inj $i = n-1$

$$\Longrightarrow \quad H^i(X, \mathcal{O}_X(-H)) = 0$$

for $i < n$

<u>Step 2</u> (Cyclic covers) $\mathcal{L}$ line bundle

$D = V(s)$ $\qquad s : \mathcal{O}_X \to \mathcal{L}^{\otimes m}$ nonzero

$$\mathcal{A}(\mathcal{L}, s) = \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}$$

$$\mathcal{L}^{-i} \otimes \mathcal{L}^{-j} \xrightarrow{\cong} \mathcal{L}^{-i-j} \xrightarrow{s \otimes id} \mathcal{L}^{m-i-j} \subseteq \mathcal{A}(\mathcal{L}, s)$$

$$X_{m,D} := \operatorname{Spec}_X \mathcal{A}(\mathcal{L}, s) \circlearrowleft \mu_m$$

locally $\mathcal{A}(\mathcal{L}, s)|_U = \mathcal{O}_U[z] / (z^m - s)$

$p : X_{m,D} \to X$ m-fold cyclic cover that is totally ramified over D

$p^* D = m D_m$ $\quad p^* \mathcal{L} \cong \mathcal{O}_X(D_m)$

$p : D_m \xrightarrow{\sim} D$ $\quad p_* D_m = D$

$p_* \mathcal{O}_{X_{m,D}} = \mathcal{A}(\mathcal{L}, s)$

(X, D) smooth $\Rightarrow (X_{m,D}, D_m)$ is a smooth pair

Step 3 (vanishing upstairs on a cyclic cover)

(X, L) as in the hypothesis

pick $D \in |mL|$ smooth

$$\begin{array}{ccc} X' := & X_{m,D} & \xrightarrow{p} X \\ \cup\!\!\mid & \cup\!\!\mid & \\ D' = & D_m & \end{array}$$

$p^* L \sim D'$

$D' \cong D$ smooth ample

X' smooth so by step 1

$$0 = H^i(X', \mathcal{O}_{X'}(-D')) = H^i(X', p^*\mathcal{O}_X(-L))$$

for $i < n$

__Step 4__ p is finite, $R^i p_* = 0$ $i > 0$

for any locally free $\mathcal{E}$

$$H^i(X', p^*\mathcal{E}) = H^i(X, p_* p^* \mathcal{E}) = H^i(X, \mathcal{E} \otimes p_*\mathcal{O}_{X'})$$

$$= \bigoplus_{k=0}^{m-1} H^i(X, \mathcal{E}(-kL))$$

$\mathcal{E} = \mathcal{O}_X(-L)$ by __step 3__ the

LHS $= 0$ $\Longrightarrow$ $H^i(X, \mathcal{O}_X(-L)) = 0$

for $i < n$ $k = 0$ $i < n$

__Thm__ (Kodaira-Akizuki-Nakano Vanishing)

X smooth projective, $\mathcal{L}$ ample line bundle then

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0 \quad \text{for } p+q > n$$

$p = n$ Kodaira

$$\Updownarrow \text{SD}$$

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^{-1}) = 0 \quad \text{for } p+q < n$$

$P = 0$ Kodaira

Pf sketch

same strategy but use

1) $0 \to \Omega_X^p \to \Omega_X^p(\log D) \xrightarrow{\text{residue}} \Omega_D^{p-1} \to 0$

$D = V(z_1) \quad \langle \frac{dz_1}{z}, dz_2, \ldots, dz_n \rangle$

2) $0 \to \Omega_X^p(\log D)(-D) \to \Omega_X^p \xrightarrow{\text{res}} \Omega_X^p \to 0$

3) $p^* \Omega_X^p(\log D) = \Omega_{X_{m}, D_m}(\log D_m)$